When $\theta = 0$; $r = p_0$, $u = u_0$, and $z = 1$, therefore $c = 0$. Consequently the equation of the curved ray may be written as

$$\frac{1}{r} = \frac{1}{p_0(hp_0 + 2l)} \left\{ l + (hp_0 + l) \cos \left[\frac{\sqrt{hp_0(hp_0 + 2l)}}{hp_0 + l} \theta \right] \right\} . \quad (7)$$

We shall next apply equation (7) to the special case of the gravitational field associated with the sun. The point S of the figure may now be considered as situated at the center of this star.

According to Einstein's theory of gravitation the index of refraction at the point P is given by $n = 1 + 2m/r$ to the first order of approximation in m/r. As a matter of fact, m^2/r^2 would be negligible in the case of the sun since, for this mass, $m = 1.473$ km., $p_0 = 6964 \times 10^2$ km. (corresponding to the sun's limb) and $m^2/p_0^2 = 4.474 \times 10^{-12}$. Consequently we may obtain an approximate equation of the path of a wave of light in the sun's gravitational field by neglecting all powers of m/r higher than the first in equation (7), and by putting $h = 1$ and $l = 2m$. [*Vide* relation (5).]

Under these conditions equation (7) reduces to

$$\frac{1}{r} = \frac{2m}{p_0^2} + \frac{p_0 - 2m}{p_0^2} \cos \theta. \quad (8)$$

This is the familiar equation of one branch of an hyperbola referred to the nearer focus as pole. The eccentricity of the hyperbola is given by $e = p_0/2m - 1$ which, in the practical case under consideration, has the very large value 2364×10^2. The transverse semi-axis $= (2mp_0)/(p_0 - 4m) = 2.946$ km., approximately.

For an infinite value of r equation (8) shows that θ is obtuse and that the approximate value of the total angular deviation, $(2D_\infty)$, from asymptote to asymptote, equals $4m/p_0$. (*Vide supra.*)

In order to calculate the deviations at various points along the curved ray it will be necessary to derive a special formula, since the numerical values of the deviations are so extremely small as to preclude the use of equation (8) in conjunction with ordinary logarithmic-trigonometric tables.

This may be accomplished in the following manner. Let

$$a \equiv \frac{p}{r} = \frac{p_0 + 2m}{r + 2m}, \qquad b \equiv \frac{p_0^2 - 2mr}{r(p_0 - 2m)}.$$

By referring to the figure and equation (8) we readily see that

$$D = \theta - \phi = \cos^{-1} b - \cos^{-1} a,$$

or

$$D = \sin^{-1} a - \sin^{-1} b,$$

or

$$D = \sin^{-1} [a \sqrt{1 - b^2} - b \sqrt{1 - a^2}]. \quad (9)$$

To the first order in m/p_0 and m/r we obtain

$$a \sqrt{1 - b^2} = c \sqrt{1 - \frac{p_0^2}{r^2}}, \qquad b \sqrt{1 - a^2} = \left(c - \frac{2m}{p_0} \right) \sqrt{1 - \frac{p_0^2}{r^2}},$$

where

$$c \equiv \frac{p_0 + 2m}{r} - \frac{2mp_0^2}{r^2(r + p_0)} .$$

Hence, relation (9) reduces to

$$D = \frac{2m}{p_0}\sqrt{1 - \left(\frac{p_0}{r}\right)^2} = \frac{1296 \times 10^3\, m}{\pi p_0 r}\sqrt{(r - p_0)(r + p_0)}'' . \qquad (10)$$

The first form of relation (10) shows that the deviation very rapidly approaches the asymptotic limit $2m/p_0$ as r increases, and the second form is especially convenient for logarithmic computation.

In calculating the data collected in the table given below the values 1.473 km. and 6964×10^2 km. were taken for m and p_0, respectively. Thus the table pertains to a ray that has grazed the sun's limb. For integral values, θ was taken as the independent variable and r was calculated from equation (8). The numbers given for r in the table have been rounded off. The angles tabulated in the right hand column were obtained by the aid of formula (10).

$\theta°$	r km.	D''
0	6964×10^2	0
5	6991×10^2	0.07605
10	7071×10^2	0.15152
15	7210×10^2	0.22584
20	7411×10^2	0.29844
25	7684×10^2	0.36876
30	8041×10^2	0.43628
35	8501×10^2	0.50048
40	9091×10^2	0.56087
45	9849×10^2	0.61700
50	1083×10^3	0.66842
55	1214×10^3	0.71476
60	1393×10^3	0.75566
65	1648×10^3	0.79081
70	2036×10^3	0.81994
75	2691×10^3	0.84283
80	4010×10^3	0.85931
85	7990×10^3	0.86925
θ_M	5794×10^4	0.87250
θ_E	1495×10^5	0.87256
90	1646×10^8	0.87257
θ_∞	∞	0.87257

The fourth and third rows from the bottom of the table pertain respectively to the orbits of the planets Mercury and Earth. $\theta_M = 89° 18' 41.5''$, $\theta_E = 89°$

$44' 0''$, and $\theta_\infty = 90° + D_\infty$. The data in the last four rows fulfill the expectation that the curved ray has sensibly reached its asymptote not only for the earth's orbit but even for an observer at the distance of the inferior planet.[1]

GRAPHICAL DISCUSSION OF THE ROOTS OF A QUARTIC EQUATION.

By E. L. REES, University of Kentucky.

It is the purpose of this note to give a graphical study of the conditions which determine the nature of the roots of a quartic equation. Using the reduced form $f(x) = x^4 + qx^2 + rx + s = 0$, with q, r and s real and with discriminant Δ, we have types of the quartic for which the following are criteria[2] regarding the nature of the roots:

$\Delta < 0$, roots distinct, two real, two imaginary;

$\Delta > 0$, roots distinct, all real or all imaginary;

$q < 0$, $s > \frac{q^2}{4}$, roots imaginary;

$s < \frac{q^2}{4}$, roots real;

$q \geqq 0$, roots imaginary;

$\Delta = 0$, at least two equal roots;

$q < 0$, $s > \frac{q^2}{4}$, two equal real roots, two imaginary;

$-\frac{q^2}{12} < s < \frac{q^2}{4}$, roots real, two and only two equal;

$s = \frac{q^2}{4}$, two pairs of equal real roots;

$s = -\frac{q^2}{12}$, roots real, three equal;

$q > 0$, $s > 0$, $r \neq 0$, two equal real roots, two imaginary;

$s = \frac{q^2}{4}$, $r = 0$, two pairs of equal imaginary roots;

$s = 0$, two equal real roots, two imaginary;

$q = 0$, $s > 0$, two equal real roots, two imaginary;

$s = 0$, four equal real roots.

The discriminant is the product of the squares of the differences of the roots

[1] The deviation of a ray from its original direction at the star is obtained by adding $2m/p_0$ to the value of D given in the table, and the total deviation, $1.745''$, from asymptote to asymptote is twice the value of D given in the last two lines of the table.—EDITORS.

[2] Compare L. E. Dickson, *Elementary Theory of Equations*, 1914, p. 45.

of $f(x) = 0$. We prove first the following theorem[1] which gives a geometric interpretation of the discriminant that will be used later in the discussion.

The discriminant of a real quartic equation $f(x) = 0$ with leading coefficient unity equals 256 times the product of the ordinates of the turning points[2] of the graph of $y = f(x)$.

Denoting by α_i or α_j the roots of $f(x) = 0$, and by α_k' those of $f'(x) = 0$, we have

$$\Delta = \prod_i \prod_{j>i} (\alpha_i - \alpha_j)^2 = \prod_i \prod_j (\alpha_i - \alpha_j) = \prod_i f'(\alpha_i)$$
$$= 4^4 \prod_i \prod_k (\alpha_i - \alpha_k') = 4^4 \prod_k \prod_i (\alpha_k' - \alpha_i) = 4^4 \prod_k f(\alpha_k').$$

Since $f(\alpha_k')$ are the ordinates of the turning points of $y = f(x)$, the proof is complete. We may now state as a corollary the following results:

$\Delta < 0$, *ordinates all negative;*
one ordinate negative, two positive;
one ordinate negative, two imaginary;
$\Delta > 0$, *ordinates all positive;*
one ordinate positive, two negative;
one ordinate positive, two imaginary;
$\Delta = 0$, *at least one ordinate zero.*

Assume the graph of $y = f(x)$ drawn. Let a line start in the position of the x-axis and revolve about the origin into the position of the line $y = -rx$, and let each point of the graph move in a vertical line at such a rate that its distance measured vertically from the revolving line remains constant.[3] The resulting curve is symmetric with respect to the y-axis and is the graph of the equation $y = x^4 + qx^2 + s$, which we shall call the auxiliary quartic. The roots of $f(x) = 0$ are the abscissas of the points of intersection of the line and this curve. The turning points of the original curve correspond to those points of the transformed curve at which the tangent is parallel to the line $y = -rx$. We shall call these points of the graph of the auxiliary quartic *transformed turning points.* The inflection points of one curve correspond to the inflection points of the other curve. The proofs of these statements are quite simple and are left to the reader.

By the usual calculus method we easily deduce the following facts concerning the auxiliary quartic curve.

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[1] The corresponding theorem for the n-ic $x^n + c_1x^{n-1} + \cdots + c_n = 0$ (c's real) is

$$\Delta = (-1)^{\frac{n(n-1)}{2}} n^n \cdot (\textit{product of ordinates of turning points}).$$

[2] In this discussion we shall understand that a point of inflection with horizontal tangent is to be considered a multiple turning point.

[3] This of course is the same as adding the ordinates of the line $y = -rx$ and the curve $y = f(x)$. The effect of this transformation in our discussion is to replace the original quartic by the auxiliary quartic curve while the line $y = -rx$ in its relation to the graph of the auxiliary quartic in a certain sense replaces the x-axis.

The three turning points[1] are $(0, s)$, $\left(\pm\sqrt{-\frac{q}{2}}, s - \frac{q^2}{4}\right)$ and the two inflection points are $\left(\pm\sqrt{-\frac{q}{6}}, s - \frac{5}{36}q^2\right)$.

If $q < 0$, these turning points and inflection points are all real and distinct.

The difference between the ordinate of the middle turning point and the common ordinate of the other two is $q^2/4$.

The distance between the y-intercept point of the curve and that of the inflection tangents $= q^2/12$.

The y-intercept of the inflection tangents is $s + q^2/12 = I$.

If $q = 0$, the y-intercept point of the curve is a triple turning point (double inflection point).

If $q > 0$, there is only one real turning point; the inflection tangents are imaginary and meet inside the curve at the point $(0, I)$; and there is a real double tangent $y = s - q^2/4$, with conjugate imaginary points of contact.

We shall study the various cases corresponding to the different forms and positions of the auxiliary quartic curve according to the sign of q and the value of s.

The corollary of the theorem on discriminants proved above may now be restated in a slightly different and more useful form, namely:

At least one and possibly three real transformed turning points lie above or below the line $y = -rx$ according as Δ is greater than or less than 0; and at least one transformed turning point (real with one exception[2]) lies on the line if $\Delta = 0$.

Bearing this corollary in mind, an examination of a figure in each case will make clear the following classification of quartics, equivalent to that given at the beginning of this paper.

Case I. $q < 0$,

$s > \frac{q^2}{4}$, $\Delta < 0$, two roots real and distinct;

$\Delta = 0$, two roots real and equal;

$\Delta > 0$, no real root;

$s = \frac{q^2}{4}$, $\Delta < 0$, two roots real and distinct;

$\Delta = 0$, two pairs of equal real roots;

$0 < s < \frac{q^2}{4}$, $\Delta < 0$, two roots real and distinct;

$s = 0$, $\Delta = 0$, all roots real, two equal;

$-\frac{q^2}{12} < s < 0$, $\Delta > 0$, all roots real and distinct;

[1] Not the "transformed turning points" mentioned in the preceding paragraph, but the turning points of the auxiliary quartic itself.

[2] It will be seen (next foot-note) that there is one case when $\Delta = 0$ in which a real turning point (the other two being imaginary) is not on the line.

$s = -\frac{q^2}{12}$, $\Delta < 0$, two roots real and distinct;

$\Delta = 0$, all roots real, three equal;

$s < -\frac{q^2}{12}$, $\Delta < 0$, two roots real and distinct.

Case II. $q = 0$,

$s > 0$, $\Delta < 0$, two roots real and distinct;
$\Delta = 0$, two roots real and equal;
$\Delta > 0$, no real roots;

$s = 0$, $\Delta < 0$, two roots real and distinct;
$\Delta = 0$, four equal real roots;

$s < 0$, $\Delta < 0$, two roots real and distinct.

Case III. $q > 0$,

$s > 0$, $\Delta < 0$, two roots real and distinct;
$\Delta = 0$, two roots real and equal;[1]
$\Delta > 0$, no real root;

$s = 0$, $\Delta < 0$, two roots real and distinct;
$\Delta = 0$, two roots real and equal;

$s < 0$, $\Delta < 0$, two roots real and distinct.

If $I < 0$, then $s < -q^2/12$ and the quartic has two real and distinct and two imaginary roots.

For a triple root it is necessary and sufficient that the line $y = -rx$ be an inflection tangent. I being the y-intercept of the inflection tangents, it follows that $I = 0$ is a necessary condition for a triple root; and since the slopes of the inflection tangents are $\pm\frac{4q}{3}\sqrt{-\frac{q}{6}}$ one of which must equal $-r$, it results that $8q^3 + 27r^2 = 0$ is also a necessary condition.

Conversely, if $I = 0$ and $8q^3 + 27r^2 = 0$, the quartic has a triple root.

Noting that

$$J\left(= \frac{qs}{6} - \frac{r^2}{16} - \frac{q^3}{216}\right) = \frac{q}{6}I - \frac{8q^3 + 27r^2}{432}$$

we see that these results are equivalent to the following familiar theorem:

A necessary and sufficient condition that the quartic equation have three or more roots equal is $I = J = 0$.[2]

The geometric arguments for most of the cases of the theorem may be made without the use of the derivative. For this purpose we apply the transformation

[1] Except when $s = q^2/4$ and $r = 0$, in which case there are two conjugate imaginary double roots.

[2] The method of this paper enables us also to recognize the order of succession of the simple and multiple roots. Thus by noting the double root cases in which the inflexions of the transformed quartic both lie below the line $y = -rx$ we find as a necessary and sufficient condition for a double root separating two simple real roots $\Delta = 0$, $8q^3 + 27r^2 < 0$.

In the other cases where all roots are real and there is one double or triple root, this will be the greatest or least root according as r is positive or negative.—EDITOR.

$x' = x^2$, $y = y$ to the parabola $y = x'^2 + qx' + s$, thus obtaining the auxiliary quartic curve. Note that the vertex of the parabola is $(-q/2, s - q^2/4)$ and that the parabola and quartic have in common the y-intercept s.

The different forms of the quartic curve depend on the position of the parabola relative to the y-axis. The three cases follow.

$q < 0$, the vertex of the parabola is to right of the y-axis; the quartic has three real and distinct turning points, $(0, s)$, $\left(\pm\sqrt{-\frac{q}{2}}, s - \frac{q^2}{4}\right)$.

$q = 0$, the vertex of the parabola is on the y-axis; the quartic has a triple turning point $(0, s)$.

$q > 0$, the vertex of the parabola is to left of the y-axis; the quartic has only one real turning point $(0, s)$.

The arguments for the various cases of the theorem with few exceptions are identical with those sketched above.[1]

TWO NEW CONSTRUCTIONS OF THE STROPHOID.

By R. M. MATHEWS, Wesleyan University.

(Read before the American Mathematical Society December 28, 1920.)

1. The classic construction for the strophoid uses a pencil of circles each of which has its center on a "medial" line g and passes through a fixed point, the node O, on g (Fig. 1). Let each circle be cut by that one of its diameters which passes through a fixed point, the singular focus F. The curve is the locus of these intersections.[2] The object of this note is to make this construction more general for the same curve: first, by using any line through the node as locus for the centers of the circles; and second, by using a pencil of circles through any two conjugate points of the curve. In preparation for this we describe certain well known features of the curve.[3]

[1] Instead of adding the ordinates of the line $y = -rx$ and the curve $y = f(x)$, the author might have started with the curve $y = x^4 + qx^2 + s$ and regarded the roots of the given quartic as the abscissas of the intersections of this curve and the line $y = -rx$. The form of this curve depends only on q; its position, or the position of the origin with respect to it, depends on s, while the character of the roots of the equation, when q and s are given, depends on r. Thus the classification, based first on q, and then on s, would finally be based on r.

The range of values of r for any type of equation, when q and s are given, depends on those values which correspond to the real tangents from the origin. These values of r are the roots of the equation $\Delta = 0$, and for any particular type of equation Δ will have a particular sign or be zero. Conversely, the sign or vanishing of Δ, with the given values of q and s, will usually determine the type of the equation. These considerations would enable us to dispense with the author's theorem on discriminants. Results obtained as depending on r could be interpreted at once as depending on Δ, and so when the classification has been obtained, the various classes could be grouped and arranged with respect to Δ, q and s if such an arrangement is more convenient for use.—EDITOR.

[2] Gino Loria, *Spezielle algebraische und transcendente ebene Kurven*, volume 1, Leipzig, 1910, p. 60. The strophoid of our text-books is the *right* strophoid, the form this curve takes when the node is at the foot of the perpendicular from the focus to the median.

[3] Loria, *loc. cit.*, chapter 8.

2. The strophoid is *the* orthotomic circular cubic. With the tangents at the node as axes, its equation may be written

$$(x^2 + y^2)(y + cx) - axy = 0;$$

or in parametric form

$$x = \frac{am}{(1 + m^2)(m + c)}, \qquad y = mx. \tag{1}$$

The real asymptote is parallel to the medial line g: $y + cx = 0$; while the two imaginary asymptotes meet at the singular focus F which is on the line $y - cx = 0$. the *axis* of the curve.

The nodal tangents bisect the angles formed by the axis and the median. Two points, P and P_1, whose parameters are m and $-m$, are "conjugate" points; that is, the tangents at these points meet the curve at the same "tangential point" T. The medial line bisects the join of every pair of conjugate points. Evidently, the nodal tangents bisect the angles determined at the node by each pair of conjugate points.

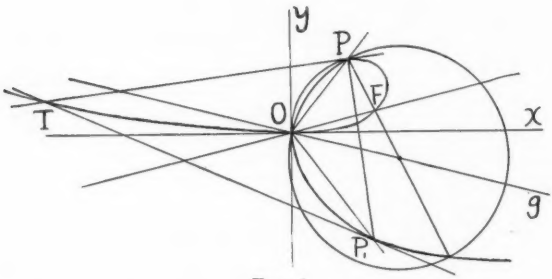


FIG. 1.

On substituting the parametric values of x and y in the equation $ux + vy + 1 = 0$ we find that the necessary and sufficient condition that three points m_1, m_2, m_3 of the strophoid be collinear is $m_1m_2m_3 = -c$.

3. Let us consider the pencil of circles of parameter h which are specified by the equation

$$x^2 + y^2 - 2hx - 2lhy = 0. \tag{2}$$

Each circle passes through the node O (Fig. 2), and has its center (h, lh) on the line $y = lx$, which we may suppose to be the line OP. When this equation is solved simultaneously with that of the strophoid, we obtain, besides the node counted twice and the circular points at infinity, the points whose parameters are roots of the equation

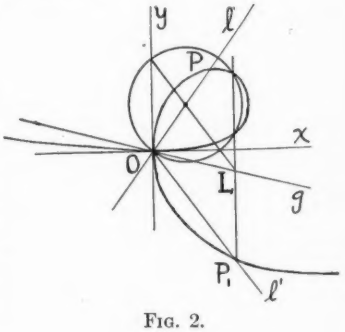


FIG. 2.

$$m^2 + \frac{1}{2lh}(2h + 2lhc - a)m + \frac{c}{l} = 0.$$

Hence $m_1m_2 = c/l$, a relation which implies $m_1m_2(-l) = -c$, and shows that

the points of intersection are collinear with the point P_1 which is conjugate to P. Thus the curve appears as the intersection of a pencil of lines with a projective pencil of circles. To correlate the two forms, we find that the join of m_1 and m_2 cuts the medial line $y + cx = 0$ in the point L

$$L: \qquad x = \frac{2lh}{l - c}, \qquad y = \frac{-2clh}{l - c}.$$

This result justifies the following construction of the curve, given the node O and two conjugate points P, P_1. Construct the tangents at the node and the medial line. Take the line OP for the line $y = lx$. The circle (2) through O with its center on this line cuts the axes again in $(2h, 0)$ and $(0, 2lh)$. The join of these points cuts the medial line in the point L, and the line P_1L cuts the circle in the desired points of the strophoid.

If we take l as the medial line, then P_1 is the singular focus and we have the classical construction.

4. Another construction can be obtained by considering the pencil of circles through two distinct conjugate points P and P_1 of parameters m and $-m$. A circle

$$x^2 + y^2 - 2hx - 2ky + d = 0 \tag{3}$$

cuts the strophoid in the circular points at infinity and at the points whose parameters are roots of the equation

$dm^4 + 2(dc - ak)m^3 + (d + dc^2 + a^2 - 2ah - 2ack)m^2 + 2c(d - ah)m + dc^2 = 0$.

Thus the necessary and sufficient condition that four points be concyclic is

$$m_1m_2m_3m_4 = c^2.$$

Taking m and $-m$ for m_1 and m_2 we have the pencil of circles on PP_1 such that

$$m_3m_4(m^2/c) = -c.$$

Therefore, every circle through the conjugate points of parameters m, $-m$ cuts the strophoid in two points Q, Q_1 which are collinear with a certain point K of parameter m^2/c. (Fig. 3.)

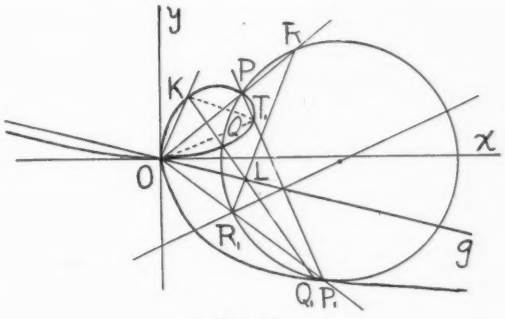


FIG. 3.

From the parametric equations it is easy to show that the slope of PP_1 is $-m^2/c$. The perpendicular from the origin upon this line meets it at the point c/m^2 of the strophoid, say T_1 (for it is the conjugate to the point T, the common tangential of P and P_1). The point at infinity of the strophoid has for parameter $-c$ and is collinear

with K and T_1, since $(m^2/c)(c/m^2)(-c) = -c$. Therefore to construct K we draw OT_1 perpendicular to PP_1, OK the reflection of OT_1 around the bisector of the angle between the nodal tangents, and T_1K parallel to the medial line g.

It remains to put the pencil of circles on PP_1 into graphical correspondence with the pencil of lines at K. Each circle through P and P_1 cuts the nodal radii OP and OP_1 again in two points R and R_1. To determine R, substitute $y = mx$ in (3). The result is a quadratic in x the product of whose roots is $d/(1 + m^2)$. But one of these is the x in (1); therefore the other root is $d(m + c)/am$. In this way we find the coördinates of R and R_1 to be:

$$\left[\frac{d}{am}(m+c), \quad \frac{d}{a}(m+c)\right] \quad \text{and} \quad \left[\frac{d}{am}(m-c), \quad \frac{-d}{a}(m-c)\right].$$

The equation of the line RR_1 is then

$$am^2x - acy - d(m^2 - c^2) = 0,$$

and it requires only some algebraic drudgery to show that this line meets KQQ_1 on the medial line g.

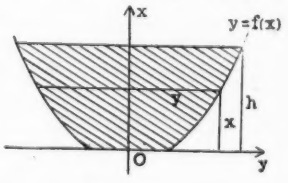
Accordingly, the strophoid may be constructed as follows, given the node and two conjugate points P and P_1. Construct the nodal tangents, the medial line, the line OK and the point K where it will cut the curve. Each circle of the pencil through PP_1 cuts the nodal radii OP, OP_1 in two points R, R_1; the line RR_1 cuts the medial line in a point L, and the line LK cuts the circle in its remaining real intersections with the strophoid.

5. While these constructions are not superior to the classical one in case of actual use on the drawing board, they are of importance as bases for the study of new properties of the curve.

AN APPLICATION OF ABEL'S INTEGRAL EQUATION.

By W. C. BRENKE, University of Nebraska.

Let the shaded area in the figure represent the cross section of a weir notch, the cross section being symmetrical with respect to the x-axis. The quantity of flow through the notch per unit time will be given by



$$Q = C\int_0^h \sqrt{h-x}\, f(x)dx,$$

where the form of the notch is determined by $y = f(x)$; $x \geqq 0$.

Consider the problem of determining $f(x)$ so that the quantity of flow per unit of time shall be proportional to a given power of the depth of stream; *i.e.*, $Q = k'h^m$, $m > 0$. Hence we must find $f(x)$ from an integral equation of the form

$$\int_0^h \sqrt{h-x}\, f(x)dx = kh^m. \tag{1}$$

Differentiation with respect to h gives

$$\int_0^h \frac{f(x)}{\sqrt{h-x}}\,dx = 2kmh^{m-1}, \tag{2}$$

and a solution of (2) will be a solution of (1) also. But (2) comes under the form of Abel's integral equation,[1]

$$\int_a^x \frac{f(y)dy}{(x-y)^s} = g(x), \qquad (0 < s < 1),$$

which has the continuous solution

$$f(x) = \frac{\sin s\pi}{\pi}\int_a^x \frac{g'(y)dy}{(x-y)^{1-s}},$$

provided that $g(x)$ is continuous and has a finite derivative $g'(x)$ with at most a finite number of discontinuities in the range of integration, and that $g(a) = 0$. These conditions are satisfied in the problem under consideration if $m \geqq 2$, and hence we have

$$f(x) = \frac{2km(m-1)}{\pi}\int_0^x \frac{y^{m-2}dy}{\sqrt{x-y}}; \qquad m \geqq 2.$$

Evaluation of the last integral leads to a simple closed form for $f(x)$. By making the change of variable $y = xt$ we obtain

$$\int_0^x \frac{y^{m-2}dy}{\sqrt{x-y}} = x^{m-3/2}\int_0^1 \frac{t^{m-2}dt}{\sqrt{1-t}}. \tag{3}$$

But from the theory of the gamma-function[2] we have

$$\int_0^1 \frac{t^{p-1}dt}{(1-t)^{1-q}} = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

Applying this to (3) and substituting in the equation for $f(x)$, we get

$$f(x) = \frac{2km(m-1)}{\pi}\cdot\frac{\Gamma(m-1)\Gamma(\frac{1}{2})}{\Gamma(m-\frac{1}{2})}\,x^{m-\frac{3}{2}},$$

or, since $k\Gamma(k) = \Gamma(k+1)$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$,

$$f(x) = \frac{2k\Gamma(m+1)}{\sqrt{\pi}\,\Gamma(m-\frac{1}{2})}\,x^{m-\frac{3}{2}}; \qquad m \geqq 2. \tag{4}$$

When $m = n$, where n is a positive integer $\geqq 2$, we have

$$f(x) = \frac{k}{\pi}\cdot\frac{2^n\,n!}{1\cdot 3\cdot 5\cdot\cdots(2n-3)}\,x^{n-\frac{3}{2}}. \tag{5}$$

[1] M. Bôcher, *An Introduction to the Study of Integral Equations*, Cambridge, University Press, 1909, pp. 8–9.

[2] Nielsen, *Handbuch der Theorie der Gamma-Funktion*, 1906, p. 133.

When $m = n + 1/2$, n as above, we have

$$f(x) = k \frac{1 \cdot 3 \cdot 5 \cdot \cdots (2n+1)}{2^n (n-1)!} x^{n-\frac{1}{2}}. \tag{6}$$

When $n = 2$, (5) gives the parabola, in which the quantity of flow is proportional to the square of the depth of stream, and (6) gives the triangular weir notch, with flow proportional to $h^{5/2}$.

It is easy now to show that (4) is still a solution of (1), though not necessarily continuous, for values of $m > 1/2$. To do this we put $x = ht$ in (1), which gives

$$\int_0^1 \sqrt{1-t}\, f(ht)dt = kh^{m-\frac{1}{2}}.$$

Substituting $f(ht) = ch^n t^n$ gives

$$c \int_0^1 \sqrt{1-t}\, t^n dt = kh^{m-n-\frac{1}{2}}.$$

The left member of this equation is independent of h, hence we must have

$$n = m - 3/2, \qquad \text{and} \qquad c \int_0^1 \sqrt{1-t}\, t^n dt = k,$$

which gives, since $\Gamma(3/2) = \sqrt{\pi}/2$,

$$c = \frac{2k}{\sqrt{\pi}} \cdot \frac{\Gamma(m+1)}{\Gamma(m-\frac{1}{2})}, \quad \text{provided } m > 1/2.$$

Hence (4) is a solution if $m > 1/2$.

When $m = 3/2$ we get the rectangular notch, $y =$ constant, and when $m = 1$ we get the curve $y = 1/\sqrt{x}$, such that the flow is directly proportional to the depth of stream.

DEPRECIATION BY A CONSTANT PERCENTAGE PLUS A CONSTANT.

By C. R. FORSYTH, Dartmouth College.

There are two ways in which a piece of property may depreciate which are usually considered in any complete treatise on the mathematical theory of depreciation, treatments in which possible interest accumulations are given no consideration. The methods employed to compute the annual or periodic allowance for depreciation corresponding to these two ways are known familiarly as the "straight line" method and the "constant percentage of book value" method. The annual allowance corresponding to the first case is the constant

$$k = \frac{C - S}{n}, \tag{1}$$

where C denotes the original cost, S the scrap value and n the estimated lifetime

of the property. The constant percentage in the second case is

$$x = 1 - \sqrt[n]{\frac{S}{C}}. \tag{2}$$

It is not the purpose of this paper to dwell upon the practical virtues of the application of the first method or the practical defects of the application of the second method but rather, since both methods occupy a fairly important place in the mathematical theory of depreciation from the point of view of the mathematical theory itself, to present results analogous to those given above when a piece of property depreciates by a constant percentage plus a constant.

If we denote the original cost by C, the constant percentage by x and the constant by k, then the amount to be charged off the first year is $Cx + k$ leaving a residue of $C(1 - x) - k$. Continuing this reasoning it is easily shown that the residue R_n at the end of the nth year will be

$$R_n = (1 - x)^n \left(C + \frac{k}{x} \right) - \frac{k}{x}. \tag{3}$$

Suppose that the valuation engineer is able to give at least three estimates R_n, R_{2n}, R_{3n}, etc., of the value of the property corresponding to the ends of at least three intervals of n years. Assuming first R_n to be known, equation (3) is easily solved for k to give

$$k = x \frac{R_n - C(1 - x)^n}{(1 - x)^n - 1}. \tag{4}$$

It remains then to determine x. Assuming now that R_{2n} and R_{3n} are also known, we may write

$$R_n = (1-x)^n \left(C + \frac{k}{x} \right) - \frac{k}{x}, \qquad R_{2n} = (1-x)^{2n} \left(C + \frac{k}{x} \right) - \frac{k}{x},$$

$$R_{3n} = (1-x)^{3n} \left(C + \frac{k}{x} \right) - \frac{k}{x}.$$

Subtracting and dividing as indicated on the left side of the following equation we obtain

$$\frac{R_{2n} - R_{3n}}{R_n - R_{2n}} = (1 - x)^n,$$

whence

$$x = 1 - \sqrt[n]{\frac{R_{2n} - R_{3n}}{R_n - R_{2n}}}. \tag{5}$$

As a simple illustration, suppose that a property depreciates annually (that is, $n = 1$) by a constant percentage plus a constant in accordance with the following data: $C = \$1000.00$, $R_1 = \$880.00$, $R_2 = \$772.00$, $R_3 = \$674.80$. Then, by formula (5)

$$x = 1 - \frac{772.00 - 674.80}{880.00 - 772.00} = 1/10.$$

By formula (4)

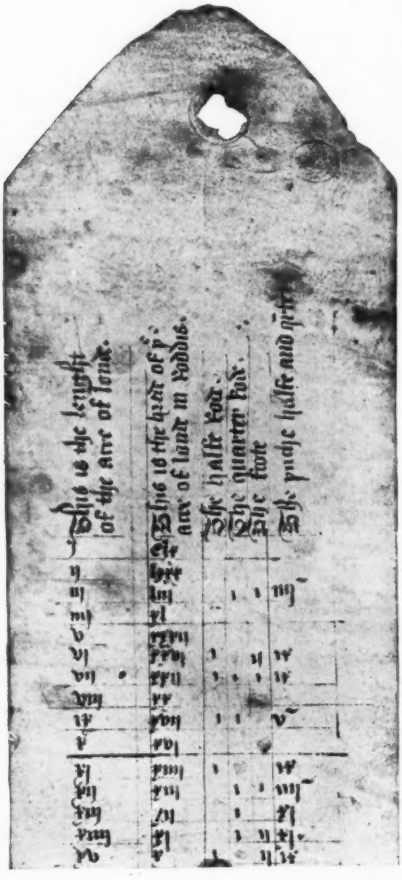
$$k = \frac{1}{10} \frac{\$880.00 - (9/10)\$1000.00}{9/10 - 1} = \$20.00.$$

As a special case, if ever $x = 0$ formula (4) takes an indeterminate form which, however, is easily evaluated in the usual way to be (1), $k = (C - S)/n$, where periodic estimates are no longer needed and R_n is replaced by S. Likewise, if $k = 0$ formula (3) becomes $S = C(1 - x)^n$, where, again, periodic estimates are no longer needed and R_n is replaced by S. Solving this equation for x we obtain formula (2).

AN INTERESTING FOURTEENTH CENTURY TABLE.

By DAVID EUGENE SMITH, Columbia University.

There has recently come into the possession of the library of Columbia University an interesting mathematical roll written apparently in the south of England about the close of the fourteenth century. It is $3\frac{9}{16}$ inches wide and $38\frac{3}{16}$ inches long and consists of two strips of parchment sewed together, only the first part of the roll being shown in the facsimile. The style of the script, the spelling, and the forms of the numerals suggest as the approximate date the year 1400. This is rather early for a mathematical manuscript in the English language, although we have others of still earlier date. English manuscripts of a mathematical nature written before the fifteenth century usually concern the interests of the less scholarly class, and, since this particular specimen relates to farm measurement, it would have been of little service had it appeared in the Latin of the church schools.



The roll consists of a table showing the widths corresponding to various lengths of a rectangular piece of land containing an acre. As the facsimile shows, the first column gives the lengths, beginning with 1 rod, the caption reading: "This is the lenght of the acre of londe." The lengths are given for every rod from 1 to 160.

The second column gives the largest number of rods in the width of the rectangular piece: "This is the brede of y^{e} acre of londe in Roddis,"—in which the symbol y^{e} is, as is well known to scholars, the old Anglo-Saxon form of "the," the letter resembling a small *y* (*th*) having been retained (as in this manuscript) long after the form *Th* (for the capital) was adopted.

The third and fourth columns give the fractions of a rod in halves and fourths, "The halfe Rode" and "The quarter Rode" forming an interesting relic of the ancient use of those unit fractions which seem to have been first employed in Mesopotamia in the third millennium B.C. and which are familiar in Egyptian mathematics for a period of upwards of two thousand years.

Instead of then proceeding to eighths, which would have been a little too difficult for the farm owner, the maker of the table gives, in the next column, the number of feet, or, as he writes it, "The foote." The final column gives the inches and fractions,—"The ynche halfe and qr̄ter."

It will be observed that the numerals are Roman, the Hindu-Arabic forms not having as yet reached the great mass of commercial and industrial people in any country north of Italy, if, indeed, they may be said to have done so in Italy itself. The Roman numerals answered all ordinary needs so long as integers alone were involved, but they failed when elaborate fractions were demanded. This manuscript shows how the difficulty was met, both by the use of the unit fractions one half and one fourth with respect to the rod, as shown by the numerator 1 in the third and fourth columns, and with respect to the inch, as shown by the characters (∼) for one half and (·) for one fourth.

The need for such a table will become apparent if the reader will endeavor to find the width of a rectangle, say 12 rods long and containing precisely one acre, making use only of Roman numerals or the abacus in his computations. In this special case the table gives as the result 13 (xiij) rods, $\frac{1}{4}$ rod, 1 foot, $4\frac{1}{2}$ (iiij∼) inches, which the reader may care to verify. The computations are accurate to a fraction of an inch, but the only fractions used in this connection are $\frac{1}{2}$ (∼), $\frac{1}{4}$ (·), and $\frac{3}{4}$ (·̃), and in some of his results the computer has erred by more than a quarter of an inch,—an error of no practical significance in the kind of computation for which the table was intended.

RECENT PUBLICATIONS.

REVIEWS.

The Absolute Relations of Time and Space. By A. A. Robb. Cambridge, at the University Press, 1921. 8vo. 9 + 80 pages. Price 5 shillings.

Preface: "At the meeting of the British Association in 1902, Lord Rayleigh gave a paper entitled 'Does motion through the ether cause double refraction?' in which he described some experiments which seemed to indicate that the answer was in the negative. I recollect that on this occasion Professor Larmor was asked whether he would expect any such effect and he replied that he did not expect any.

"In the discussion which followed reference was made to the null results of all attempts to detect uniform motion through the aether and to the way in which things seemed to conspire together to give these null results.

"The impression made on me by this discussion was: that in order properly to understand what happened, it would be necessary to be quite clear as to what we mean by equality of lengths, etc., and I decided that I should try at some future time to carry out an analysis of this subject.

"I am not certain that I had not some idea of doing this even before the British Association meeting, but in any case, the inspiration came from Sir Joseph Larmor, either at this meeting or on some previous occasion while attending his lectures.

"Some years later I proceeded to try to carry out this idea, and while engaged in endeavouring to solve the problem, I heard for the first time of Einstein's work.

"From the first I felt that Einstein's standpoint and method of treatment were unsatisfactory, though his mathematical transformations might be sound enough, and I decided to proceed in my own way in search of a suitable basis for a theory.

"In particular I felt strongly repelled by the idea that events could be simultaneous to one person and not simultaneous to another; which was one of Einstein's chief contentions.

"This seemed to destroy all sense of the reality of the external world and to leave the physical universe no better than a dream, or rather, a nightmare.

"If two physicists A and B agree to discuss a physical experiment, their agreement implies that they admit, in some sense, a common world in which the experiment is supposed to take place.

"It might be urged perhaps that we have merely got a correspondence between the physical worlds of A and B, but if so, where, or how, does this correspondence subsist?

"It cannot be in A's mind alone, or it would not be a correspondence, and similarly it cannot be in B's mind alone.

"It seems to follow that it must be in some common sub-stratum; and this brings us at once back to an objective standpoint.

"The first work which I published on this subject was a short tract entitled *Optical Geometry of Motion, a New View of the Theory of Relativity* which appeared in 1911.

"This paper, though it did not claim to give a complete logical analysis of the subject, yet contained some of the germs of my later work and, in particular, it avoided any attempt to identify instants at different places. Later on the idea of *'Conical Order'* occurred to me, in which such instants are treated as definitely distinct.

"The working out of this idea was a somewhat lengthy task and in 1913 I published a short preliminary account of it under the title *A Theory of Time and Space*, which was also the title of a book on this subject on which I was then engaged.

"This book was in the press at the time of the outbreak of the war and was finally published toward the end of 1914.

"Unhappily at that period people were concerning themselves rather with trying to sever one another's connexions with Time and Space altogether, than with any attempt to understand such things; so that it was hardly an ideal occasion to bring out a book on the subject.

"The subject moreover was not an easy one, and I have been told more than once that my book is difficult reading.

"To this I can only reply as did Mr. Oliver Heaviside, under similar circumstances, that it was perhaps even more difficult to write.

"Be that as it may, the results arrived at fully justified my attitude towards Einstein's standpoint.

"I succeeded in developing a theory of Time and Space in terms of the relations of *before* and *after*, but in which these relations are regarded as absolute and not dependent on the particular observer.

"In fact it is not a 'theory of relativity' at all in Einstein's sense, although it certainly does involve relations.

"These relations of *before* and *after*, serving, as they do, as a physical basis for the mathematical theory, were quite ignored in Einstein's treatment; with the result that the absolute features were lost sight of.

"Even now, some six years from the date of publication of my book, comparatively few of Einstein's followers appear to realize the extreme importance of these relations, or to recognize how they alter the entire aspect of the subject.

"The theory, in so far as its postulates have an interpretation, becomes a physical theory in the ordinary sense, but these postulates are used to build up a pure mathematical structure.

"From the physical standpoint the question is: whether the postulates *as interpreted* are correct expressions of physical facts, or in some respect only approximations?

"If the postulates are not all correct expressions of the facts, then which of them require emendation and what emendation do they require?

"As regards the pure mathematical aspect of the theory: this of course remains unaffected by the physical interpretation of the postulates, and those who are interested only in pure mathematics may find that the method employed has certain advantages as a study of the foundations of geometry.

"In particular it may be noticed that by this method we get a system of geometry in which 'congruence' appears, not as something extraneous grafted on to an otherwise complete system, but as an intrinsic part of the system itself.

"I had intended making further developments of this theory, but the outbreak of the war caused an interruption of my work.

"In the meantime Einstein produced his 'generalized relativity' theory and the reader will doubtless wish to know how this work bears upon it.

"So far as I can at present judge, the situation is this: once coördinates have been introduced, the theory here developed gives rise to the same analysis as Einstein's so-called 'restricted relativity' and this latter cannot be regarded as satisfactory apart from my work, or some equivalent.

"Einstein's more recent work is extremely analytical in character.

"The *before* and *after* relations have not been employed at all in its foundation, although it is evident that, if these relations are a sufficient basis for the simple theory, they must play an equally important part in any generalization. Moreover these relations most certainly have a physical significance whatever theory be the correct one.

"A generalization of my own work is evidently possible and, to a certain extent, I can see a method of carrying this out, although I have not as yet worked out the details. (See Appendix.)

"In the meantime it seemed desirable to write some sort of introduction to my *Theory of Time and Space* which, while not going into the proofs of theorems, would yet convey to a larger circle of readers the main results arrived at in that work."

Contents—Preliminary considerations, 1–16; Conical order, 16–45; Normality of general lines having a common element, 46–55; Theory of congruences, 56–71; Introduction of coördinates, 72–75; Interpretation of results, 76–78; Appendix, 78–80.

Introduction to the Theory of Fourier's Series and Integrals. By H. S. CARSLAW. Second edition, completely revised. London, Macmillan, 1921. 8vo. 11 + 323 pp. Price 30 shillings.

Preface: "This book forms the first volume of the new edition of my book on *Fourier's Series and Integrals and the Mathematical Theory of the Conduction of Heat*, published in 1906, and now for some time out of print. Since 1906 so much advance has been made in the Theory of Fourier's Series and Integrals, as well as in the mathematical discussion of Heat Conduction, that it has seemed advisable to write a completely new work, and to issue the same in two volumes. The first volume, which now appears, is concerned with the Theory of Infinite Series and Integrals, with special reference to Fourier's Series and Integrals. The second volume will be devoted to the Mathematical Theory of the Conduction of Heat.

"No one can properly understand Fourier's Series and Integrals without a knowledge of what is involved in the convergence of infinite series and integrals. With these questions is bound up the development of the idea of a limit and a function, and both are founded upon the modern theory of real numbers. The first three chapters deal with these matters. In Chapter IV the Definite Integral is treated from Riemann's point of view, and special attention is given to the question of the convergence of infinite integrals. The theory of series whose terms are functions of a single variable, and the theory of integrals which contain an arbitrary parameter are discussed in Chapters V and VI. It will be seen that the two theories are closely related, and can be developed on similar lines.

"The treatment of Fourier's Series in Chapter VII depends on Dirichlet's Integrals. There and elsewhere throughout the book, the Second Theorem of Mean Value will be found an essential part of the argument. In the same chapter the work of Poisson is adapted to modern standard, and a prominent place is given to Fejér's work, both in the proof of the fundamental theorem and in the discussion of the nature of the convergence of Fourier's Series. Chapter IX is devoted to Gibbs's Phenomenon, and the last chapter to Fourier's Integrals. In this chapter the works

of Pringsheim, who has greatly extended the class of functions to which Fourier's Integral Theorem applies, has been used.

"Two appendices are added. The first deals with 'Practical Harmonic Analysis and Periodogram Analysis.' In the second a bibliography of the subject is given.

"The functions treated in this book are 'ordinary' functions. An interval (a, b) for which $f(x)$ is defined can be broken up into a finite number of open partial intervals, in each of which the function is monotonic. If infinities occur in the range, they are isolated and finite in number. Such functions will satisfy most of the demands of the Applied Mathematician.

"The modern theory of integration, associated chiefly with the name of Lebesgue, has introduced into the Theory of Fourier's Series and Integrals functions of a far more complicated nature. Various writers, notably W. H. Young, are engaged in building up a theory of these and allied series much more advanced than anything treated in this book. These developments are in the meantime chiefly interesting to the Pure Mathematician specialising in the Theory of Functions of a Real Variable. My purpose has been to remove some of the difficulties of the Applied Mathematician."

Contents—Historical introduction, 1–15; Chapter I: Rational and irrational numbers, 16–28; II: Infinite sequences and series, 29–48; III: Functions of a single variable. Limits and continuity, 49–75; IV: The definite integral, 76–121; V: The theory of infinite series whose terms are functions of a single variable, 122–168; VI: Definite integrals containing an arbitrary parameter, 169–195; VII: Fourier's series, 196–247; VIII: The nature of the convergence of Fourier's series, 248–263; IX: The approximation curves and Gibbs's phenomenon in Fourier's series, 264–282; X: Fourier's integrals, 283–294; Appendix I: Practical harmonic analysis and periodogram analysis, 295–301; II: Bibliography, 302–317; List of authors quoted, 318–319; General index, 320–323.

Each chapter concludes with bibliographical "References." The names of the following Americans occur in the work: BÔCHER, BYERLY, FORD, GRONWALL, JACKSON, C. N. MOORE, and VAN VLECK.

Plane and Solid Analytic Geometry. By W. F. OSGOOD and W. C. GRAUSTEIN. New York, The Macmillan Company, 1921. 12mo. 17 + 614 pp. Price $3.75.

Preface: "The object of an elementary college course in Analytic Geometry is twofold: it is to acquaint the student with new and interesting and important geometrical material, and to provide him with powerful tools for the study, not only of geometry and pure mathematics, but in no less measure of physics in the broadest sense of the term, including engineering.

"To attain this object, the geometrical material should be presented in the simplest and most concrete form, with emphasis on the geometrical content, and illustrated, whenever possible, by its relation to physics. This principle has been observed throughout the book. Thus, in treating the ellipse, the methods actually used in the drafting room for drawing an ellipse from the data commonly met in descriptive geometry are given a leading place. The theorem that the tangent makes equal angles with the focal radii is proved mechanically: a rope which passes through a pulley has its ends tied at the foci and is drawn taut by a line fastened to the pulley. Moreover, the meaning of foci in optics and acoustics is clearly set forth. Again, there is a chapter on the deformations of an elastic plane under stress, with indications as to the three-dimensional case (pure strain, etc.).

"The methods of analytic geometry, even in their simplest forms, make severe demands on the student's ability to comprehend the reasoning of higher mathematics. Consequently, in presenting them for the first time, purely algebraic difficulties, such as are caused by literal coefficients and long formal computations, should be avoided. The authors have followed this principle consistently, beginning each new subject of the early chapters with the discussion of a simple special, but typical, case, and giving immediately at the close of the paragraph simple examples of the same sort. They have not, however, stopped here, but through carefully graded problems, both of geometric and of analytic character, have led the student to the more difficult applications of the methods, and collections of examples at the close of the chapters contain such as put to the test the initiative and originality of the best students.

"As a result of this plan the presentation is extraordinarily elastic. It is possible to make the treatment of any given topic brief without rendering the treatment of later topics unintelligible, and thus the instructor can work out a course of any desired extent. For example, one freshman

course at Harvard devotes about thirty periods to analytic geometry and the material consists of the essential parts of the first nine chapters. Another freshman course gives twice the time to analytic geometry (the students having already had trigonometry), taking up determinants and the descriptive properties of the quadric surfaces, and also devoting more time to the less elementary applications of the methods of analytic geometry. The advanced courses in the calculus and mechanics require the material of the later chapters. In fact, a thorough elementary treatment of the rudiments of Solid Analytic Geometry is indispensable for the understanding of standard texts on applied mathematics. It is true that these texts are chiefly Continental. But we shall never have American treatises which are up to the best scientific standards of the day until the subjects above mentioned are available in simply intelligible form for the undergraduate.

"The subject of loci is brought in early through a brief introductory chapter, and problems in loci are spread throughout the book. A later chapter is devoted to a careful explanation of the method of auxiliary variables. There is a chapter on determinants, with applications both to analytic geometry and to linear equations. Diameters and poles and polars in the plane and in space receive a thorough treatment. Cylindrical and spherical coördinates and quadric surfaces are illumined by the concept of triply orthogonal systems of surfaces. The reduction of the general equation of the second degree in space to normal forms by translations and rotations is sketched and illustrated by numerical examples.

"The question may be asked: In so extensive a treatment of analytic geometry should not, for example, homogeneous coördinates find a place? The authors believe that the student, before proceeding to the elaborate methods of modern geometry, should have a thorough knowledge both of the material and the methods which may fairly be called elementary, and they felt that a book which, avoiding the conciseness of some of the current texts and the looseness of others, is clear because it is rigorous will meet a real need.

"This book is designed to be at once an introduction to the subject and a handbook of the elements. May it serve alike the needs of the future specialist in geometry, the analyst, the mathematical physicist, and the engineer."

Contents. *Plane analytic geometry*—Introduction: Directed line-segments. Projections, 1–6; Chapter I: Coördinates. Curves and equations, 7–26; II: The straight line, 27–52; III: Applications, 53–64; IV: The circle, 65–78; V: Introductory problems in loci. Symmetry of curves, 79–87; VI: The parabola, 88–100; VII: The ellipse, 101–123; VIII: The hyperbola, 124–153; IX: Certain general methods, 154–192; X: Polar coördinates, 193–215; XI: Transformation of coördinates, 216–234; XII: The general equation of the second degree, 235–260; XIII: A second chapter on loci. Auxiliary variables. Inequalities, 261–287; XIV: Diameters. Poles and polars, 288–329; XV: Transformations of the plane. Strain, 330–359; XVI: Determinants and their applications, 360–404. *Solid analytic geometry*—Chapter XVII: Projections. Coördinates, 405–420; XVIII: Direction cosines. Direction components, 420–443; XIX: The plane, 444–469; XX: The straight line, 470–497; XXI: The plane and the straight line. Advanced methods, 498–522; XXII: Spheres, cylinders, cones. Surfaces of revolution, 523–547; XXIII: Quadric surfaces, 548–583; XXIV: Spherical and cylindrical coördinates. Transformation of coördinates, 584–606; Index, 607–614.

Einstein's Theories of Relativity and Gravitation. A Selection of Material from the Essays submitted in the Competition for the Eugene Higgins Prize of $5,000. Compiled and edited, and introductory matter supplied, by J. M. BIRD, associate editor of the *Scientific American*. New York, Scientific American Publishing Co., 1921. 12mo. 14 + 345 pp. Price $2.00.

We have already referred [*1921*, 191–192; 269] to the prize offered by the *Scientific American*, to the conditions of its award, and to essays presented in competition for it. In the volume under review the prize essay is not given till pages 169–180 and this is followed by a dozen other essays. It was designed by the editor that "Each essay should be made easier of reading by the examination of those preceding it; at the same time each, by the choice of ground covered and by the emphasis on points not brought out sharply by its predecessors, should throw new light upon these predecessors." Chapters II–VI, pages 19–

168, are given up to the discussion of various ideas for making the reading of the essays more profitable; for this purpose, in three of these chapters, the general plan has been to make extracts from about fifty other of the competing essays, the extracts being connected by editorial comment. For example, in the chapter on "The special theory of relativity" there are extracts from competing essays of twenty-six authors, interlarded with more than twenty-five editorial comments. The following is the complete table of contents:

Preface, iii–xii; Chapter I: The Einstein $5,000 prize: how the contest came to be held, and some of the details of its conduct, by the editor, 1–18; II: The world—and us: an introductory discussion of the philosophy of relativity, and of the mechanism of our contact with time and space, by various contributors and the editor, 19–46; III: The relativity of uniform motion: classical ideas on the subject; the ether and the apparent possibility of absolute motion; the Michelson-Morley experiment and the final negation of this possibility, by various contributors and the editor, 47–75; IV: The special theory of relativity: what Einstein's study of uniform motion tells us about time and space and the nature of the external reality, by various contributors and the editor, 76–110; V: The parallel postulate: modern geometric methods; the dividing line between Euclidean and non-euclidean; and the significance of the latter, by the editor, 111–140; VI: The space-time continuum: Minkowski's world of events, and the way in which it fits into Einstein's structure, by the editor and a few contributors, 141–168; VII: Relativity: the winning essay in the contest for the Eugene Higgins $5,000 prize, by LYNDON BOLTON, British Patent Office, London, 169–180; VIII: The new concepts of time and space: the essay in behalf of which the greatest number of dissenting opinions have been recorded, by MONTGOMERY FRANCIS, New York, 181–194; IX: The principle of relativity: a statement of what it is all about, in ideas of one syllable, by HUGH ELLIOT, Chiselhurst, Kent, England, 195–205; X: Space, time and gravitation: an outline of Einstein's theory of general relativity, by W. DE SITTER, University of Leyden, 206–217; XI: The principle of general relativity: how Einstein, to a degree never before equalled, isolates the external reality from the observer's contribution, by E. T. BELL, University of Washington, 218–229; XII: Force vs. geometry: how Einstein has substituted the second for the first in connection with the cause of gravitation, by SAUL DUSHMAN, Schenectady, 230–239; XIII: An introduction to relativity: a treatment in which the mathematical connections of Einstein's work are brought out more strongly and more successfully than usual in a popular explanation, by HAROLD T. DAVIS, University of Wisconsin, 240–250; XIV: New concepts for old: what the world looks like after Einstein has had his way with it, by JOHN G. MCHARDY, Commander R. N., London, 251–264; XV: The new world: a universe in which geometry takes the place of physics, and curvature that of force, by GEORGE FREDERICK HEMENS, M.C., B.Sc., London, 265–275; XVI: The quest of the absolute: modern developments in theoretical physics, and the climax supplied by Einstein, by Dr. FRANCIS D. MURNAGHAN [*1921*, 269], Johns Hopkins University, Baltimore. 276–286; XVII: The physical side of relativity: the immediate contacts between Einstein's theories and current physics and astronomy, by Professor WILLIAM H. PICKERING, Harvard College Observatory, Mandeville, Jamaica, 287–305; XVIII: The practical significance of relativity: the best discussion of the special theory among all the competing essays, by Prof. HENRY NORRIS RUSSELL, Princeton University, 306–317; XIX: Einstein's theory of relativity: a simple explanation of his postulates and their consequences, by T. ROYDS, Kodaikanal Observatory, India, 318–326; XX: Einstein's theory of gravitation: the discussion of the general theory and its most important application, from the essay by Prof. W. F. G. SWANN, University of Minnesota, Minneapolis, 327–333; XXI: The equivalence hypothesis: the discussion of this, with its difficulties and the manner in which Einstein has resolved them, from the Essay by Prof. E. N. DA C. ANDRADE, Ordnance College, Woolwich, England, 334–337; XXII: The general theory: fragments of particular merit on this phase of the subject, by various contributors, 338–345.

An Introduction to Mathematical Analysis. By F. L. GRIFFIN. Boston, Houghton, Mifflin Company, 1921. 12mo. 8 + 512 pp. Price $2.75.

Extracts from the Preface: "Under the traditional plan of studying trigonometry, college algebra, analytic geometry, and calculus separately, a student can form no conception of the character and possibilities of modern mathematics, nor of the relations of its several branches as parts of a unified whole, until he has taken several successive courses. Nor can he, early enough,

get the elementary working knowledge of mathematical analysis, *including integral calculus*, which is rapidly becoming indispensable for students of the natural and social sciences. Moreover, he must deal with complicated technique in each introductory course; and must study many topics apart from their uses in other subjects, thus missing their full significance and gaining little facility in drawing upon one subject for help in another.

"To avoid these disadvantages of the separate-subject plan the unified course presented here has been evolved. This enables even those students who can take only one semester's work to get some idea of differential and integral calculus, trigonometry, and logarithms. And specialist students, as experience has shown, acquire an excellent command of mathematical tools by first getting a bird's-eye view of the field, and then proceeding to perfect their technique.

"A regular course in calculus, following this, can proceed more rapidly than usual, include more advanced topics, and give a fine grasp: the principles and processes have become an old story. And the regular course in analytic geometry can be devoted to a genuine study of the geometrical properties of loci, since most of the type equations, basic formulas, and calculus methods are already familiar.

"The materials presented here have been thoroughly tried out with the freshman classes in Reed College during the past nine years. Problems and methods which have proved unsatisfactory have been eliminated. Care has been taken to make the concepts tangible, relate them to the familiar ideas of daily life, exhibit practical applications, and develop the attitude of investigation. . . .

"The course as given at Reed College takes four hours a week through the year, the number of lessons devoted to the several chapters, when taken complete, having run about as follows: 14, 4, 14, 8, 11, 12, 11, 16, 5, 7, 10, 6, 6, 5, 4. . . . The course is adapted to students of widely differing preparations. A knowledge of plane and solid geometry and of algebra through quadratics is the most suitable equipment; but a number of students who had had only two years of secondary mathematics have carried the course very well. On the other hand, students who have already taken trigonometry and college algebra find in the present course very little that merely duplicates their former work."

Contents—A preliminary word to students, 1–2; Chapter I: Functions and graphs (Some fundamental problems of variation: rates, mean values, extremes, zero values, formulas, etc.), 3–57; II: Some basic ideas analyzed (Instantaneous rates, tangents, areas, etc., as limits), 58–75; III: Differentiation (Derivatives of polynomials and u^n. Rates, extremes, etc.), 76–125; IV: Integration ($\int x^n dx$. Area, volume, momentum, work, fluid pressure, falling bodies, etc.), 126–155; V: Trigonometric functions (Solution of right and oblique triangles. Applications), 156–188; VI: Logarithms (Numerical calculations. Compound interest. Triangles), 189–235; VII: Logarithmic and exponential functions, 236–270; VIII: Rectangular coördinates (Mapping. Motion. *Analytic geometry:* line, circle, parabola, ellipse, hyperbola; translation, intersections), 271–325; IX: Solution of equations (Quadratics: $b^2 - 4ac$. Rational roots of higher equations. Horner's and Newton's methods), 326–342; X: Polar coördinates and trigonometric functions (Definitions. Radians. Periodic variations. Derivatives), 343–367; XI: Trigonometric analysis (Basic identities. Equations. More calculus. Involute. Cycloid. *S.H.M.* Damped oscillations. Addition formulas. Sums and products, etc.), 368–391; XII: Definite integrals (Summation of "elements": length, surface of revolution, etc. Plotting a surface. Double integration. Partial derivatives. Simpson's rule), 392–414; XIII: Progressions and series (*A.P.* and *G.P.* Investment theory. Maclaurin series. Calculation of functions. Binomial theorem), 415–439; XIV: Permutations, combinations and probability ($P_{n,r}$; $C_{n,r}$. Chancel Normal probability curve. Least squares), 440–459; XV: Complex number system (Definition. Geometric representation. Operations. Roots of unity. Application), 460–472; Retrospect and prospect, 472–483; Appendix (Proofs for reference. Formulas. Integrals. Numerical tables: roots, natural and common logarithms, trigonometric functions for radians or degrees), 485–508; Index, 509–512.

Analytic Geometry with Introductory Chapter on the Calculus. By C. I. PALMER and W. C. KRATHWOHL. New York, McGraw-Hill Book Co., 1921. 12mo. 14 + 347 pages. Price $2.50.

Preface: "The object of this book is to present analytic geometry to the student in as natural and simple a manner as possible without losing mathematical rigor. The average student thinks visually instead of abstractly, and it is for the average student that this work has been written.

It was prepared primarily to meet the requirements in mathematics for the second half of the first year at the Armour Institute of Technology. To make it adaptable to courses in other institutions of learning certain topics not usually taught in an engineering school have been added.

"While it is useless to claim any great originality in treatment or in the selection of subject matter, the methods and illustrations have been thoroughly tested in the class room. It is believed that the topics are so presented as to bring the ideas within the grasp of students found in classes where mathematics is a required subject. No attempt has been made to be novel only; but the best ideas and treatment have been used, no matter how often they have appeared in other works on the subject.

"The following points are to be especially noted:

(1) The great central idea is the passing from the geometric to the analytic and *vice versa*. This idea is held consistently throughout the book.

(2) In the beginning a broad foundation is laid in the algebraic treatment of geometric ideas. Here the student should acquire the analytic method if he is to make a success of the course.

(3) Transformation of coördinates is given early and used frequently throughout the book, not confined to a single chapter as is so frequently the case. The same may be said of polar coordinates.

(4) Fundamental concepts are dealt with in an informal as well as in a formal manner. The informal often fixes and clarifies the ideas where the formal does not.

(5) Numerous illustrative examples are worked out in order that the student may get a clear idea of the methods to be used in the solution of problems.

(6) The conic sections are treated from the starting point of the *focus* and *directrix* definition.

(7) Because of its great importance in engineering practice the empirical equation is dealt with more completely than is usual. This treatment has been made as elementary as possible, but sufficiently comprehensive to enable one to solve the average problem in empirical equations.

(8) The fundamental concepts of the calculus are presented in a very concrete manner, and a much greater use than is usual is made of the differential. The ideas are thus more readily visualized than is possible otherwise. The applications are mainly to tangents, normals, areas, and the discussion of equations.

(9) The concluding chapter gives an adequate and careful treatment of solid geometry so necessary in the study of the calculus.

(10) The exercises are numerous, carefully graded, and include many practical applications.

(11) In the introductory chapter are found various short tables and formulas, and at the end are given four-place tables of logarithms and trigonometric functions."

Contents—Chapter I: Introduction, 1–7; II: Geometric facts expressed analytically, and conversely, 8–43; III: Loci and equations, 44–58; IV: The straight line and the general equation of the first degree, 59–85; V: The circle and certain forms of the second degree equation, 86–97; VI: The parabola and certain forms of the second degree equation, 98–116; VII: The ellipse and certain forms of the second degree equation, 117–133; VIII: The hyperbola and certain forms of the second degree equation, 134–153; IX: Other loci and equations, 154–187; X: Empirical loci and equations, 188–205; XI: Poles, polars, and diameters, 206–215; XII: Elements of calculus, 216–260; XIII: Solid analytic geometry, 261–303; Summary of formulas, 303–306; Four-place table of logarithms, 308–309; Table of trigonometric functions, 310–314; Answers, 315–340; Index, 341–347.

Plane Trigonometry. By ARNOLD DRESDEN. New York, John Wiley & Sons, 1921. 8vo. 7 + 110 pages. Price $1.60.

From the Preface: "While the importance of the function concept for elementary mathematics has become recognized by many writers of college algebra texts and of 'unified freshman mathematics' books, it has received little recognition from writers on elementary trigonometry. To emphasize this importance has been the leading motive in writing the present book. A somewhat detailed study of the graphs of the trigonometric functions (Chapter V) and of the inverse functions (Chapter VIII) has been introduced for this purpose. Much more could and should be done in this direction; perhaps the present effort may suffice as a first step.

"The opportunity afforded by the writing of a new text has been used to make some changes in the presentation of the traditional material. Circular measurement of angles is introduced in the first chapter so as to be available for use throughout the course. The fundamental theorems on projections are presented early and are used subsequently so that the student may be familiar

with them when they are applied in a general proof of the addition theorems, based on a method quite generally followed by continental writers. Recognizing the value of the 'solution of triangles,' a good deal of space has been devoted to this subject, and an attempt has been made to develop it in such a manner that the students can appreciate the reasons for the different methods that are discussed.

"On the question of 'applied problems,' I have taken a definite position. I do not think it feasible to introduce into an elementary text technical material from applied sciences, important though such material may be. Without such material, however, applications cannot well be anything but problems which use the language of the applied sciences without really belonging to them. An elementary text can render useful service, even to applied science, by stressing the fundamental concepts of trigonometry and by setting problems which connect with the student's actual experience and which suggest ways in which these concepts may be applied, leaving actual applications to the fields to which they belong.

"It has not seemed desirable to add to the number of tables of logarithms already available. The elementary treatment of logarithms in Chapter III and the problems scattered throughout the book call for the use of a set of five-place tables, of which there are many excellent ones in existence.

"No attempt at logical completeness has been made, but rather has it been my aim to adapt the treatment to the stage of logical development which may be expected of students who begin the study of trigonometry. I am aware of the fact that a fuller discussion might be made in several instances and I shall be happy if the treatment as given should arouse the critical powers of some students and develop in them a desire for more penetrating analysis.

"The material as here presented was used originally in mimeographed form by a few classes in the University of Wisconsin."

Contents—Chapter I: Positive and negative lines and angles. Coördinates. Radian measurement, 1–8; II: The trigonometric ratios. Simple identities, 9–19; III: Logarithms, 20–31; IV: Solution of right triangles. Applications, 32–40; V: The graphs of the trigonometric functions, 41–54; VI: The addition formulæ, 55–64; VII: The solution of triangles, 65–89; VIII: Inverse trigonometric functions. Trigonometric equations, 90–103; List of answers to the exercises, 105–108; Index, 109–110.

Elements of Map Projection with Applications to Map and Chart Construction. By C. H. DEETZ and O. S. ADAMS. (Department of Commerce, U. S. Coast and Geodetic Survey, serial no. 146, special publication no. 68.) Washington, Government Printing Office, 1921. Royal 8vo. 163 pp. + 8 plates. Price $.50.

Preface: "In this publication it has been the aim of the authors to present in simple form some of the ideas that lie at the foundation of the subject of map projections. Many people, even people of education and culture, have rather hazy notions of what is meant by a map projection, to say nothing of the knowledge of the practical construction of such a projection.

"The two parts of the publication are intended to meet the needs of such people; the first part treats the theoretical side in a form that is as simple as the authors could make it; the second part attacks the subject of the practical construction of some of the most important projections, the aim of the authors being to give such detailed directions as are necessary to present the matter in a clear and simple manner.

"Some ideas and principles lying at the foundation of the subject, both theoretical and practical, are from the very nature of the case somewhat complicated, and it is a difficult matter to state them in a simple manner. The theory forms an important part of the differential geometry of surfaces, and it can only be fully appreciated by one familiar with the ideas of that branch of science. Fortunately, enough of the theory can be given in simple form to enable one to get a clear notion of what is meant by a map projection and enough directions for the construction can be given to aid one in the practical development of even the more complicated projections.

"It is hoped that this publication may meet the needs of people along both of the lines indicated above and that it may be found of some interest to those who may already have a thorough grasp of the subject as a whole."

Contents—*Part I:* General statement, 7–8; Analysis of the basic elements of map projection, 9–21; Representation of the sphere upon a plane, 22–29; Elementary discussion of various forms

of projection, 30–52. *Part II:* Introduction, 53–57; The polyconic projection, 58–66; The Bonne projection, 67–70; The Lambert zenithal (or azimuthal) equal-area projection, 71–76; The Lambert conformal conic projection with two standard parallels, 77–86; The Grid system of military mapping, 87–90; The Albers conical equal-area projection with two standard parallels, 91–100; The Mercator projection, 101–136; Fixing position by wireless directional bearings, 137–139; The gnomonic projection, 140–145. *World maps:* The Mercator projection; The stereographic projection; The Aitoff equal-area projection of the sphere; The Mollweide homalographic projection; Goode's homalographic projection (interrupted) for the continents and oceans; Lambert projection of the northern and southern hemispheres; Conformal projection of the sphere within a two-cusped epicycloid; Guyou's doubly periodic projection of the sphere, 146–160. Index, 161–163.

Examples in Differential and Integral Calculus with Answers. By the late C. S. JACKSON. (Longmans' Modern Mathematical Series.) London and New York, Longmans, Green and Co., 1921. 8vo. 8 + 142 pages. Price $3.25.

First paragraphs of preface by W. M. Roberts: "This collection of Examples in the Calculus, which was made by the late Mr. C. S. Jackson, should have been published in 1917. Mr. Jackson's sudden and regrettable death in October, 1916 [see this MONTHLY *1917*, 144] caused the publication to be delayed till after the War. The book is hampered by having to be put through the press by other hands than the author's.

"A great many of those examples which can be classed as problems were constructed by Mr. Jackson himself in connection with his work at the Royal Military Academy, Woolwich, and the many public examinations in which he took part. Many of them are very neat applications of the Calculus to practical problems, and it is hoped that these will prove particularly useful to teachers who require, in their work, a number of examples which are not mere Algebraical manipulations. Many books on the Calculus treat the subject chiefly as an extension of Algebra and Analytical Geometry. This collection should be a useful supplement to such books."

Contents—*Part I, Differential Calculus*, 1–53: Differentiation; tangents and slopes; dy/dx as a rate of increase; easy maxima and minima; velocity; differentiation of logarithms; errors and rates; approximation to roots of equations; Newton's method of approximating to the roots of an equation; maxima, minima; harder questions on tangents and normals; errors; velocity; miscellaneous examples; successive differentiation; the theorem of Leibnitz; miscellaneous expansions; indeterminate forms; Taylor's theorem and applications; curvature; examples on maps; partial differentiation. *Part II, Integral Calculus*, 54–120: Known results of differentiation; methods of integration; hyperbolic functions; integration; areas; planimeters and integraphs; areas and volumes; volumes; problems on simple integration; mean values; rectification and areas of surfaces; centers of gravity; second moments, or moments of inertia; center of pressure; pendulum; differential equations; double and triple integration. *Answers*, 121–142.

This work is one of the Series containing G. B. Mathews's *Projective Geometry* (1914), Hilda P. Hudson's *Ruler and Compasses* (1916), H. S. Carslaw's *Elements of Non-Euclidean Geometry* (1916), and H. Bateman's *Differential Equations* (1918).

Higher Mechanics. By HORACE LAMB. Cambridge, at the University Press, 1920. 8vo. 10 + 272 pages. Price 21 shillings.

Preface: "This book treats of three-dimensional Kinematics, Statics, and Dynamics in what is I think a natural, as I have found it to be a convenient, order. It may be regarded as a sequel to two former treatises[1] to which occasional reference is made; but it is not dependent on these, and will I trust be readily followed by students who are conversant with ordinary two-dimensional Mechanics.

"The subject is of course a very wide one, and some principle of selection is necessary. I have tried to confine myself to matters of genuine kinematical or dynamical importance, avoiding developments whose interest, often considerable, is purely mathematical or now mainly historical. It is owing to such considerations that whilst some account is given of the Theory of Screws, of Null-Systems, and of Least Action, on the other hand brachistochrone problems, and the general theory of the Differential Equations of Dynamics, are left untouched.

"The book does not claim to be more than an elementary one, regard being had to the nature

[1] "*Statics*, Cambridge, 1912, and *Dynamics*, Cambridge, 1914."

of the subject. The reader who wishes to carry his studies further will find ample assistance in Thomson and Tait, in Rayleigh's *Theory of Sound*, and in Whittaker's *Analytical Dynamics*. And in common with other recent writers I must mention with a special sense of obligation the works of Routh, which in their later forms are an almost inexhaustible storehouse of theorems and results, and abound in interesting historical references."

Contents—Chapter I: Kinematics of a rigid body. Finite displacements, 1–13; II: Infinitesimal displacements, 14–33; III: Statics, 34–65; IV: Moments of inertia, 66–73; V: Instantaneous motion of a body (kinematics), 74–88; VI: Dynamical equations, 89–111; VII: Free rotation of a rigid body, 112–128; VIII: Gyrostatic problems, 129–150; IX: Moving axes, 151–176; X: Generalized equations of motion, 177–207; XI: Theory of vibrations, 208–248; XII: Variational methods, 249–270; Index, 271–272.

ARTICLES IN CURRENT PERIODICALS.

AMERICAN JOURNAL OF SCIENCE, fifth series, volume 2, September, 1921: "Some mechanical curiosities connected with the earth's field of force" by W. D. Lambert,[1] 129–158 [List of mechanical curiosities: " (1) The limiting level surface with the sharp edge. (2) The smooth lake with different elevations for its two ends. (3) The tendency of large bodies to 'fall' towards the equator. (4) The tendency of a rod suspended horizontally like the Eötvös balance to 'fall' by twisting about the supporting fiber. (5) The existence of great local irregularities in the curvature of the level surfaces and the interesting possibilities that the study of these irregularities seems likely to offer. "]

ANNALES SCIENTIFIQUES DE L'ECOLE NORMALE SUPÉRIEURE, third series, volume 38, August to December, 1921: "Problèmes d'hydrodynamiques relatifs aux mouvements glissants" by R. Thiry, 229–339; "Sur les ensembles abstraits" by M. Fréchet, 341–388; "Recherches sur le théorème de M. Picard" by G. Valiron, 389–429; "Rectification et complément au mémoire de la goutte liquide tournante" by J. Boussinesq, 431–437 (see *1922*, 20).

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, volume 27, June–July, 1921: "The April meeting of the American Mathematical Society" by R. G. D. Richardson, 389–400; "The Easter meeting of the Society at Chicago" by A. Dresden, 401–410; "The April meeting of the San Francisco Section" by B. A. Bernstein, 411–414; "The general theory of approximation by polynomials and trigonometric sums" by D. Jackson, 415–431; "The Einstein solar field" by L. P. Eisenhart, 432–434; "A covariant of three circles" by A. B. Coble, 434–437; "On skew parabolas" by Mary F. Curtis, 437–438; "The dispersion of observations" by J. L. Coolidge, 439–442; "The isomorphisms of complex algebra" by N. Wiener, 443–445; "On the generalization of certain fundamental formulas of the mathematical theory of finance" by C. H. Forsyth, 446–452; "The spread of Newtonian and Leibnizian notations of the calculus" by F. Cajori, 453–458; "Group theory reviews in the *Jahrbuch über die Fortschritte der Mathematik*" by G. A. Miller, 459–462; "Recent books on vector analysis" by J. B. Shaw, 463–465 [reviews of R. Gans's *Einführung in die Vektoranalysis, mit Anwendungen auf die mathematische Physik*, fourth edition (Leipzig and Berlin, 1921), of L. Silberstein's *Elements of Vector Algebra* (New York, 1919), of C. Runge's *Vektoranalysis*. Vol. I. *Die Vektoranalysis des dreidimensionalen Raumes* (Leipzig, 1919), of R. Leveugle's *Précis de Calcul géométrique* (Paris, 1920)]; Reviews by R. D. Carmichael of *Œuvres de G. H. Halphen* (2 vols., Paris, 1916–1918), 466–468, of G. H. Hardy's *Some famous problems of the theory of numbers and in particular Waring's problem* (Oxford, 1920), 471–475, and of O. Stolz and J. A. Gmeiner's *Theoretische Arithmetik* (2d ed., 2 vols., Leipzig, 1911–1915), 485; Reviews by D. E. Smith of F. Cajori's *A History of the Conceptions of Limits and Fluxions in Great Britain from Newton to Woodhouse* (Chicago, 1919), 468–470, of P. Kirchberger's *Mathematische Streifzüge durch die Geschichte der Astronomie* (Leipzig, 1921), 479–480, and of Emma Gifford's *Natural Tangents* (Manchester, England, 1920), 480; Review by J. B. Shaw of H. Lacaze's *Cours de Cinématique Théorique* (Paris, 1920), 470; Reviews by E. B. Wilson of H. Lamb's *Statics, including Hydrostatics and the Elements of the Theory of Elasticity* (Cambridge, 1916), 475–477, and of J. R. Eccles's *Advanced Lecture Notes on Light* (Cambridge, 1919), 485–486; Reviews by C. L. E. Moore of M. d'Ocagne's *Principes usuels de Nomographie* . . . (Paris, 1920),

[1] On page 141 he shows that a rod suspended horizontally (Eötvös torsion balance of the first kind) will tend to point east and west because the meridian sections of level surfaces in the earth's field of force are more curved than prime vertical surfaces. This tendency would seem to more than balance the tendency to point north and south that is due directly to the centrifugal force according to the statement in *Nature* of October 20, page 240 (see *1922*, 22).

477–478, and of K. P. Williams's *Dynamics of the Airplane* (New York, 1921), 483; Review by H. L. Rietz of *Œuvres Complètes de Christiaan Huygens* (vol. 14, The Hague, 1920), 481–482; Review by J. W. Young of D. E. Smith's *The Sumario Compendioso of Brother Juan Diez: The Earliest Mathematical Work of the New World* (Boston, 1921), 484; Review by H. H. Mitchell of *Materialien für eine wissenschaftliche Biographie von Gauss*, Heft 8, *Zahlbegriff und Algebra bei Gauss* by A. Fraenkel (Leipzig, 1920), 486–487; Notes, 488–495; New publications, 496–499; Thirtieth annual list of papers, 500–507; Index of volume 27, 508–516.

L'EDUCATION MATHÉMATIQUE, volume 24, November 1, 1921: "La géométrie du compas" (fin.), 17–18.

ENGINEERING NEWS RECORD, volume 87, November 17, 1921: "Some weaknesses of aerial photo maps" by P. J. Barry, 828 ["The principles of astronomy, trigonometry, geometry, geodesy and perspective cannot be assigned to oblivion by merely pressing a button. It is not as easy as that."]—November 24: "Improving the terminology in mechanics" by J. H. Griffith, 865—December 1: "Triangulating under difficulties" by H. W. Bradstreet, 899–900.

JAHRESBERICHT DER DEUTSCHEN MATHEMATIKER-VEREINIGUNG, volume 30, nos. 1–4 (issued October, 1921): "Mathematik und räumliche Anschauung" by R. Baldus, 1–15 [Translated extract: The oldest definition of mathematics designated it as the theory of magnitudes. This definition is to be avoided (even if the notion of magnitude be made clear) because there are wide domains of mathematics which it fails to embrace. One need only mention analysis situs, the theory of aggregates, and projective geometry. Were one, as has been done more recently, to describe mathematics as the science of drawing logically necessary conclusions, then mathematics and logical thought would coincide; this definition extends beyond mathematics. . . . These (Euclid's 13 Books of Elements) even as works of art influenced the non-mathematician very strongly. "Euclid's *Elements* seems to me almost as beautiful as Homer's *Iliad*" said H. S. Chamberlain in his *Foundations of the Nineteenth Century*]; "Die Mathematik in der Schulreform" by W. Lietzmann, 59–68 [Section titles: 1. Grundschule; 2. Die bisherigen höheren Knabenschulen; 3. Die Ziele des mathematischen Unterrichts in den höheren Schulen; 4. Die deutsche Oberschule; 5. Die Aufbauschulen; 6. Das höhere Mädchenschulwesen; 7. Die Mittelschulen; 8. Die Gabelungstendenzen; 9. Die Lehrerbildungsfrage; 10. Fortbildungseinrichtungen.]—Supplement: "Mathematische Gesellschaft in Göttingen," 32 [November 23, 1920: "Runge, Amerikanische Arbeiten über Sternhaufen und die Milchstrasse"; February 21–22, 1921: "Hilbert, Eine neue Grundlegung des Zahlbegriffes."]; "Mathematisches Kränzchen in Prag," 32 [February 4, 1921: "Winternitz, Die Knoppsche Erzeugungsweise der Kurven von Peano, Osgood und v. Koch"]; "Einstein-Preis des Scientific American," 34 [See, *1921*, 191]—Nos. 5–8 (issued November, 1921): "W. R. Hamiltons Bedeutung für die geometrische Optik" by G. Prange, 69–82 [An appreciation and exposition of the fundamental and extensive results in geometrical optics obtained by W. R. Hamilton nearly one hundred years ago, only particular parts of which are generally familiar, such as the Hamilton-Jacobi theory in analytical mechanics, Hamilton's formula in the theory of congruences of lines in differential geometry, and the discovery of conical refraction in physics]; "Zur projektiven Differentialgeometrie der Ebene" by L. Berwald, 110–121 [A study of the "accompanying triangle," with reference to E. J. Wilczynski, S. W. Reaves, and others]—Supplement: "Programm der Jahresversammlung in Jena" (September 18–24, 1921), 45–47 [three papers on postulational treatment: "Fraenkel, Über die Zermelosche Begründung der Mengenlehre"; "Bernays, Über die Hilbertsche Grundlegung der Arithmetik"; "Hertz, Über die Minimalzahl von Axiomen für ein System von Sätzen und den Begriff des idealen Elementes."]; "Preisaufgaben und gekrönte Preisschriften," 52 [reannouncement of Wolfskehl prize of 100,000 marks "für denjenigen . . . dem es zuerst gelingt, den Beweis des grossen Fermatschen Satzes zu führen."]

JOURNAL OF THE INDIAN MATHEMATICAL SOCIETY, volume 13, August, 1921: "On the cartesian oval" (second paper) by W. R. Aiyar, 121–132; "On the expansion of certain functions (with properties of associated co-efficients)" by C. Krishnamachari, 133–146; "Short notes: Linear systems of the third order on a conic" by R. Vaidyanathaswami, 147–150; Questions and Solutions, 151–160.

MATHEMATICAL GAZETTE, volume 10, December, 1921: "A school course in surveying" by H. R. Vernon, 353–358 ["The field-work and plan-drawing are done during Geography lessons, while the calculation of area, height finding and resection are undertaken by the mathematics classes." For school children of ages 10–13, graded]; "Missing-figure problems" by W. E. H. Berwick, 359–362 [A number of new and interesting problems in "skeleton" addition, multiplication and division. See *1921*, 37, 278]; "Sign in elementary analytical geometry" by F. G. Brown,

363–368 ["It is not claimed that the conventions indicated in this brief paper are entirely satisfactory; Bôcher in his *Plane Analytic Geometry* indicates that neither his own convention 'nor any other one which could be made' can be satisfactory. . . . Immediately you try to frame satisfactory rules for the perpendicular *from a point* to a line you run up against the fundamental convention of Cartesians, which requires (x, y) to be arrived at by travelling from the axes and not towards them."]; "Gleanings far and near," 358, 362, 368 [No. 88. "The American Cocker—'I have heard that devils can be raised with Daboll's arithmetic. . . . That's my small experience as far as the Massachusetts calendar, and Bowditch's navigator, and Daboll's arithmetic go.'—p. 515, Herman Melville's *Moby Dick* (World's Classics)"]; Review of W. F. Osgood's *Elementary Calculus* by C. Jones, 381–382.

MATHEMATICS TEACHER, volume 14, May, 1921: "Why it is impossible to trisect an angle or to construct a regular polygon of 7 or 9 sides by ruler and compasses" by L. E. Dickson, 217–223; "College entrance requirements in mathematics: preliminary report of National Committee on Mathematical Requirements," 224–245; "Comments on the teaching of geometry" by F. C. Touton, 246–251; "Some ideals in teaching mathematics" by E. W. Schreiber, 252–254; "Computation in junior high school mathematics" by D. W. Werremeyer, 255–260; "The slide rule in business" by S. L. Shelley, 261–263; "A program of investigation and coöperative experimentation in the mathematics of the seventh, eighth and ninth school years" by R. Schorling and J. R. Clark, 264–275; "Testing as a means of improving the teaching of high school mathematics" by E. R. Breslich, 276–291; New Publications, 292–295—October: "The aims of mathematical education" by J. H. Minnick, 297–304; "Empirical results in the theory of numbers" by R. D. Carmichael, 305–310; "Teaching pupils how to study mathematics" (to be continued) by A. Davis, 311–320; "*La Disme* of Simon Stevin—The first book on decimals" by Vera Sanford, 321–33; "No homework for mathematics pupils" by H. C. Wright, 334–336; "The future of secondary instruction in geometry" by H. E. Webb, 337–341; "The slide rule as a subject of regular class instruction in mathematics" by W. E. Breckenridge, 342–343; News and Notes, 344–348; Book Reviews, 349–354.

NATURE, volume 108, October 27, 1921: "A system of space-time co-ordinates" by J. L. Synge, 275—November 3: "Psychological tests for vocational guidance," 321–323 [A report of a conference. "Mr. D. Kennedy Fraser (lecturer in education at the University of Edinburgh) spoke upon similar lines. He described from personal experience the use of intelligence tests in America. . . . He strongly urged the execution of similar research in this country"]; "University and educational intelligence," 323–324 ["The Rhodes Trust has issued a statement for the academic year 1920–21 dealing with the scholarships it administers. . . . 129 (Rhodes scholars were in residence) from the United States . . . mathematics had six Rhodes Scholars" (altogether)]—November 17: "Reflection 'Halo' of (semi-) cylindrical surfaces" by J. H. Shaxby, 369; "Problems of physics" by O. W. Richardson, 372–377 ["Abridged from the presidential address delivered to Section A (Mathematics and Physics) of the British Association at Edinburgh on September 9"]; "Notes," 380 ["The Copley Medal (was presented in 1921 by the Royal Society) to Sir Joseph Larmor, for his researches in mathematical physics"]—November 24: "The tendency of elongated bodies to set in the north and south direction" by E. H. Grove-Hills, 403 [A reply to Sir Arthur Shuster. "The whole matter is fully discussed in an article by Mr. W. D. Lambert of the United States Coast and Geodetic Survey in the *American Journal of Science*" (see above)]; "Societies and Academies: Royal Society, November 17," 421 ["The design of repeating patterns" by P. A. MacMahon and W. P. D. MacMahon (Quotation: "The study and classification of repeating patterns in space of two dimensions is founded upon the simplest geometrical forms which happen to be repeats. These are employed as bases and are subjected to specified transformations which depend upon certain contact systems. . . . Repeats are of three varieties, the block, the 'stencil,' and the 'archipelago.' "), "The mathematical foundations of theoretical statistics" by R. A. Fisher]; "Philosophical Society, October 31," 421 ["Convex solids in higher space" by W. Burnside; "The fifth book of Euclid's 'Elements' " by M. J. M. Hill.]

PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES, volume 7, March, 1921 (received October 1): "Normalized geometric systems" by A. A. Bennett, 84–89; "Problems of potential theory" by G. C. Evans, 89–98—May: "An overlooked infinite system of groups of order pq^2" by G. A. Miller, 146–148—June: "A formula for the viscosity of liquids" by H. B. Phillips, 172–177—September: "The average of an analytic functional" by N. Wiener, 253–260; "Semi-covariants of a general system of linear homogeneous differential equations" by E. B. Stouffer, 273–276; "An algorism for differential invariant theory" by O. E. Glenn, 276–279—

October: "The average of an analytic functional and the Brownian movement" by N. Wiener, 294–298; "An integral equation and its applications" by E. Hille, 303–305.

REVUE GÉNÉRALE DES SCIENCES, volume 32, October 15, 1921: "Application des méthodes interférentielles aux mesures astronomiques," 531–532.

SCHOOL SCIENCE AND MATHEMATICS, volume 21, December, 1921: "The value and method of the historical element in the teaching of secondary mathematics" by J. T. Vallandingham, 817–822; "Worth while work with algebra failures" by Helen I. Westley, 822–825; "Teaching proportions in geometry and algebra" by J. A. Nyberg, 868–874; "Note on prime numbers" by G. A. Miller, 874 [Giving a formula proved by Tschebychef, and then a modified form. "The main object of this note is to warn the readers thereof not to adopt the former formula, which appears in various reliable works, including Landau's *Primzahlen*, volume 1, 1909, page 22. The present writer employed this formula several years ago in Miller, Blichfeldt, Dickson *Finite Groups*, 1916, page 167, where the latter formula would have been more useful. Hence this warning is the more earnest especially since the latter is such a direct consequence of the former and the notion of prime number is so very elementary. The note may also serve to illustrate the fact that obvious improvements are sometimes overlooked by the best authors."]

SCIENCE, new series, volume 54, October 7, 1921: "A new graphic analytic method" by R. von Huhn, 334–336—October 28: "The American Mathematical Society" by R. G. D. Richardson, 416—November 11: "A notable mathematical gift" [the subsidy promised to the Mathematical Association of America by Mrs. Mary Hegeler Carus] by G. A. Miller, 456—November 25: "The eclipse expeditions to Christmas Island," 513 ["It is hoped to confirm the results obtained by the British expeditions at Principe and Sobral during the eclipse of May, 1919"]; Review by C. J. Keyser of B. Russell's *The Analysis of Mind* (New York, 1921), 518–520.

SCIENCE PROGRESS, volume 16, October, 1921: "Recent advances in pure mathematics" by Dorothy M. Wrinch, 173–178; Review by G. B. J. of H. S. Carslaw's *Introduction to the Theory of Fourier's Series and Integrals* (London, 1921), 324.

SCIENTIFIC MONTHLY, volume 14, January, 1922: "Certain unities in science" by R. D. Carmichael, 41–59; "Thomas Hariot—1560–1621" by F. V. Morley, 60–66.

SIGMA XI QUARTERLY, volume 9, September, 1921: "What should the Society of Sigma Xi preach?" by G. A. Miller, 64–66.

SPHINX-ŒDIPE, volume 16, September, 1921: "Notice sur Charles Ange Laisant" (suite) by H. Brocard, 129–133; "Conférences Math. à Bruxelles (1920–1921)," 133–134; "Tables de M. Kraïtchik," 135; "Questions et Réponses," 135–143; "Histoire de Sciences" [account of an early French periodical, *Le Géomètre*, see below], 143–144; "Divers: Télescope géant," 144—October: "Arithmetic of potentials" by T. J. Barniville, 145–147; "Sur les racines primitives relatives au mod. P^n dans un corps algébrique quelconque" by G. Métrod, 147–150; "Construction d'une polygraphie de cavalier avec Sphinx en partant d'un circuit complet et fermé" (suite et fin.) by "Un Abonné," 150–154; "Sur l'équation (1) $x^4 + ax^2y^2 + y^4 = z^2$, a positif" by M. Rignaux, 154–155; "Questions et Réponses," 156–160; "Divers: Astronomie," 160 [L'astronome américain Pickering vient de faire des observations, qui, d'après lui, prouveraient que la vie existe à la surface lunaire."]

["'Le Géomètre' était un Recueil de mathématiques, à l'usage des candidats aux Ecoles Royales Polytechnique, de St. Cyr, de la Marine des Eaux et Forêts, etc. publié par M. GUILLARD, ancien élève de l'Ec. Normale, Agrégé de l'Université, Prof. de Math. Elém. au Collège Royal de Louis le Grand, Paris, chez l'éditeur Rue St. Jacques N° 121 à partir du 15 Mars 1836. Ce volume de 224 pp. (13 x 21) et 9 pl., était composé chez Ch. Eberhart, imprimeur du Collège R. de France, 12 Rue du Foin St. Jacques, et l'on y trouve la liste de 245 souscripteurs, dont certains se sont fait un nom, depuis cette époque: GERONO, ex-principal du Collège de Lorient; STURM, prof. au Collège Rollin; le baron REYNAUD, examinateur; MIQUEL, élève de l'Institution Barbet; CATALAN, régent de Math. au Collège de Châlons-sur-Marne; Le FEBURE de FOURCY, examinateur; TERQUEM, bibliothécaire du dépôt central d'artillerie; CHASLES anc. élève de l'Éc. Polyt., à Chartres; . . . .

"Ce journal, dont nous avons ainsi les 14 premieres feuilles parues donnait des mémoires, des questions et des réponses. Je signalerai les solutions de CHAUWIN (conc. général math. elém. 1822), VANÉECHOUT (conc. général de 1809, prem. et 2e classes de math. des lycées de Paris), BRUYÈRE (conc. de 1817, Elém.), GIORGINI (1812, Spéc.), GERONO (1818, Elém.), LATOUR et de PRIVEZAC (concours de 1814, Spéciales). . . .

"A signaler aussi: un théorème remarquable de MIQUEL (p. 166) et la Quest. 3 (p. 14, rép. p. 164–166): *Diviser un triangle scalène en quatre parties égales, par 2 lignes qui se coupent à angles droits.* Cette question a été reprise en 1894 dans l'*Interm. des Math.*"]

["TÉLESCOPE GÉANT: M. MACAFEE, bien connu aux U. S. A. pour ses travaux astron., en collab. avec le Prof. DAVID TODD, de Harvard s'occupe actuellement en France d'établir les plans du plus gigantesque télescope connu. On verra Mars aussi aisément que si l'on n'était qu'à 2 km. 500 de distance.—Pour obvier aux difficultés que présente la construction d'un colossal miroir réflecteur, M. MacAfee remplace celui-ci par une cuvette de métal de 50 pieds de diamètre, remplie de mercure; un mouvement rotatif imprimé a cette cuvette lui donne la concavité voulue, et lui fait jouer l'office de réflecteur. On announce la livraison pour 1924."]

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to E. L. DODD, 3012 West Ave., Austin, Texas.

CLUB ACTIVITIES.

THE MATHEMATICS CLUB OF BROWN UNIVERSITY, Providence, R. I.

[*1918*, 33; *1919*, 167; *1920*, 28, 223.]

The officers for the session 1921–1922 are the following: Chairman, Professor R. G. D. Richardson; Committee on program, Professor Burgess, Professor Gilman, Frances Merriam, Gr., Elizabeth Stafford '23, Charles Hopkins '22, George Sauté '24; Committee on arrangements, Mr. C. D. Wentworth, instructor, Nellie Stokes '23, Frances Wright '23, Clarence Eddy '22, Phillip Welch '23.

In a printed program, the following papers are announced:

October 27, 1921: "The story of a mathematical book" by Professor D. E. Smith, of Columbia University.

December 9: "Lewis Carroll" by Dorothy Rickenbacker '23; "Number systems on other bases than ten" by Clarence Bennett '23; "Various proofs of the Theorem of Pythagoras" by George Sauté '24.

January 13, 1922: "Mathematics of the calendar" by Henry Bodwell '24; "The three point problem" by Frances Merriam, Gr.

February 17: "Skeleton division" by Clarence Eddy '22; "Cardan and the solution of the cubic" by Elizabeth Stafford '23; "The inscribed regular heptagon" by Frances Wright '23.

March 10: "Some special graphical methods" by Professor O. D. Kellogg, of Harvard University.

April 28: "Cryptography" by Donald MacPherson, Gr.; "Galois" by Katherine Colton '22; "Measuring the diameter of Betelgeuse" by Charles Hopkins '22.

May: Picnic.

MATHEMATICS CLUB OF COLUMBIA UNIVERSITY, New York.

[*1918*, 227; *1919*, 262; *1920*, 425.]

The following officers were elected for the year 1920–21: President, Albert Meder, Jr. '22; vice-president, Oscar Bodansky '22; secretary, William Thompson '23. The following papers were read:

October 1, 1920: "Some graphical methods" by Albert Meder, Jr. '22.

October 15: "Mathematical bits from Leacock" by Oscar Bodansky '22.

October 29: "Formulæ of investments" by Professor L. P. Siceloff.

November 12: "The probability equation" by A. Preisman '22.

November 26: "Constructibility" by O. Frink, Jr. '22.

December 10: "Analytic methods" by R. Kronig '22.

January 12, 1921: "Relations" by Professor C. J. Keyser.

February 18: "Volumes and areas with empirical boundaries" by W. Skeats '23.

March 4: "Convergent series" by Professor W. B. Fite.

March 18: "Measuring Betelgeuse" by M. Schwartzschild '22.

April 8: "Discontinuous functions" by Professor T. S. Fiske.

April 22: "The mathematics of wound healing" by William Thompson '23.
May 6: "Types of geometry" by Dr. K. Lamson.
The average attendance was about fifteen.

(Reported by Mr. Thompson.)

THE GRINNELL COLLEGE MATHEMATICS CLUB, Grinnell, Iowa.

[*1918*, 449.]

November 2, 1920: "Archimedes, his life and work" by Louise Pinkerton '22. Election of officers.
November 16: "Fallacies" by Elizabeth Pace '22; "Simultaneous equations" by Burton Jones '23. Problems.
November 30: "Proofs of the Pythagorean theorem" by Raymond Weigen '22.
December 6: "The mathematical treatment of certain classes of scientific data" by Professor H. L. Rietz, of The State University of Iowa. This was a joint meeting with the Science Club.
January 11, 1921: "Chinese swan pan" by Ruth Raffety '23; "Mathematics of the calendar" by Margaret Divelbess '23. The constitution presented by Louise Pinkerton '22.
March 8: "First mathematical work printed in the New World" by Ruth Boyce '22; "Plato's general attitude toward mathematics" by Frances Nelson '22.
March 22: Social meeting.
April 5: "Projective geometry" by Professor O. W. Albert.
April 19: "Newton, his life and work" by Robert Hannelly '23; "The adding machine" by Leonard Paulu '22.
May 3: "Constructions on the circle" by Frances Harshberger '23; "College as a training school for mathematics teachers" by Dorothy Rollette '23.
May 10: "Euclid's parallel postulate" by Lillie Mann '23.
May 23: Picnic. Election of officers.

(Reported by Miss Harshberger.)

UNDERGRADUATE MATHEMATICS CLUB, UNIVERSITY OF ILLINOIS, Urbana, Ill.

[*1918*, 404; *1919*, 309.]

The Club was organized under a new constitution March 24, 1920, which provided that members be undergraduates belonging to one of the following classes: (1) Students with an average of 2/3 A and 1/3 B in mathematics grades or the equivalent; (2) Juniors or seniors majoring in mathematics; (3) Students unanimously elected by a committee of the Club. There is, however, one faculty member, acting as adviser. There are no dues. The Club is for work or mathematical recreation only.

The regular meetings are from 7 to 8 p.m., weekly; and are open to the public,—the business meetings being separate. The first quarter of the hour is devoted to problems. An address, about one-half hour long, follows. Discussion concludes the program. Each member is expected to give one address during the year. The general theme for the year was "The applications of mathematics."

The average attendance is about twenty.

The following officers were elected for 1920–21: President, John Arnold '21; vice-president, Lorraine Conrad '21; secretaries, Lloyd Baker '22, Vincent Duvigneaux '22, Nelson Sowers '24.

The papers presented were:
October 6, 1920: "Applications of mathematics to chemistry" by Vincent Duvigneaux '22.
October 13: "Mathematical aspects of statistics" by Professor A. R. Crathorne.
October 20: "Applications of mathematics to insurance" by Lorraine Conrad '21.
October 27: "Applications of mathematics to physics" by Professor A. W. Williams.
November 3: "Slide rules" by Dr. H. M. Westergaard, associate.
November 10: "The influence of mathematical theories in the history of psychology" by Professor C. A. Ruckmick.
November 17: "Logarithm tables" by Professor Joel Stebbins.
November 24: "Mathematics in electricity" by Lloyd Baker '22.
December 1: "Mathematics in sound" by Dorothy Briggs '21.
December 8: "Einstein's restricted theory of relativity" by John Arnold '21.

December 15: "The physical basis of Einstein's theory" by Dr. W. H. Hyslop, instructor in Physics.
December 22: "Mathematics in electrical engineering" by A. R. Knight, of the Department of Electrical Engineering.
January 19, 1921: "Linkages" by Arthur Dixon '23.
February 22: "The importance of mathematics in genetic problems" by Professor J. A. Detlefsen, of the College of Agriculture.
March 1: "Mathematics in geology" by Professor T. T. Quirke.
March 8: "The quantitative determination of examinations" by Dr. C. R. Griffith, instructor.
March 15: "Codes and ciphers" by Major G. F. N. Dailey.
March 22: "Professor Michelson's interferometer" by John Worley '23.
April 5: "The determination of π" by Dorothy Briggs '21.
April 12: "The history of the calendar" by Nelson Sowers '24.
April 19: "Mathematical short cuts" by Arthur Dixon '23.
April 26: "Whole numbers" by Professor A. J. Kempner.
May 10: "Calculating machines" by Frank Hutchins '24.
May 17: "Mathematics in music" by John Arnold '21.

By constitutional revision May 10, 1921, the Club decided to separate into two divisions, a men's and a women's division, each self-governed, but closely federated under the old name. While both are concerned with pure mathematics, the men, in addition, will be interested more generally in engineering fields or other scientific work, while the women will be interested in teaching.

The officers of the Men's Mathematics Club for 1921–22 are: President, Vincent Duvigneaux '22; vice-president, Edgar Leach '22; secretaries, Nelson Sowers '24 and Frank Hutchins '24. The officers of the Women's Mathematics Club are: President, Doris Engle '22; vice-president, Lois Trogdon '22; secretary, Nellie Hunt '23; treasurer, Viola Judy '22.

(Reported by Mr. Sowers.)

Mathematics Club of the University of Montana, Missoula, Mont.

[*1918*, 408.]

The officers for the year 1920–21 were: President, William Walterskirchen '21; vice-president, Mayme Carney '22; secretary, Cecil Phipps '21; treasurer, Catherine Hauck '23.

The following papers were read:
October 27, 1920: "A proof of the transcendence of e" by Professor E. A. F. Carey.
November 11: "The impossibility of the trisection of the angle" by Hugh Norville '21.
December 9: "$e^{i\pi}$" by Hilda Benson '22.
January 12, 1921: "Developments from the exponential series" by Professor A. S. Merrill.
January 26: "Methods of sailing" by Harry Rooney '22.
February 23: "The fundamental concepts of relativity" by Cecil Phipps '21.
March 9: "Probability in a game of dice" by Ruth McQuay '22.
March 23: "Experimental verification of relativity" by Cecil Phipps '21.
April 27: "DeMoivre's theorem" by Kay McKoin '22.
May 25: "The simple pendulum" by William Walterskirchen '21.

(Reported by Mr. Phipps.)

Pi Mu Epsilon, University of Pennsylvania, Philadelphia, Pa.

[*1918*, 455.]

The chapter of Pi Mu Epsilon recently organized at the University of Pennsylvania is the successor to the "Vinculum." Members of the Mathematics Department, graduates, and undergraduates majoring in mathematics are eligible to membership. The organization has about thirty members, eight of whom are of the faculty. During the past year the majority of the meetings were of a purely business nature.

The officers for 1921–22 are: Director, Professor E. S. Crawley; vice-director, Professor J. H. Minnick, dean of the School of Education; secretary, Ella Rosentoor '22; treasurer, Esther Bernstein '23; librarian, Mabel Kessler '21.

The following were open meetings:

February 18, 1921: "Einstein's theory" by Professor C. Richards, of the Department of Mathematical Physics.
April 22: "Astronomy," an illustrated lecture by Professor S. G. Barton, of the Department of Astronomy.

(Reported by Miss Rosentoor.)

Pi Mu Epsilon, Syracuse University, Syracuse, N. Y.

[*1918*, 271.]

The officers for the year 1920–21 were: Director, Professor J. J. Nassau; vice-director, Professor Mary Harwood; secretary, Marion French '21; treasurer, Sterling Pugh '21; librarian, Leda Gast '21. Two of the faculty, one graduate and eight undergraduates were elected to membership. There was a Christmas party and a picnic in addition to the following meetings:
October 11, 1920: "Computation of the orbits of comets" by Professor L. Lindsay.
November 8: "History and ideals of Pi Mu Epsilon" by Professor E. D. Roe, Jr.
December 13: "Research work" by Professor W. H. Metzler; "A slide rule for the solution of quadratic equations" by Professor Nassau.
February 7, 1921: "Quaternions" by Professor Roe; "The geometry of point space of n dimensions" by Professor Nassau.
March 21: "Some theorems on determinants" by Jung Sun, Gr.
April 25: "Relativity" by Dean Graham, professor of Electrical Engineering. Open meeting.

(Reported by Professor Nassau.)

The Mathematical and Physical Society of the University of Toronto, Toronto, Ont.

[*1918*, 229; *1919*, 169.]

Officers for the session 1920–21 were elected as follows: Honorary president, Professor Lachlan Gilchrist; president, Aylmer Paisley '21; vice-president, James Phillips '22; secretary, Myra McLean '22; treasurer, William Webster '23; corresponding secretary, Eva Henry '21; representatives—fourth year, Lawrence Rentner '21; third year, Eric Horwood '22; second year, Max Bell '23; first year, Grizilda Bovard '24.

Meetings were held on alternate Thursdays at 4:15 p. m.
October 21, 1920: "How to build up atoms" by Professor J. C. McLennan.
November 4: "Physics, ancient and modern" by Percy Lowe, Gr.; "A college education for business" by Dallas Bates, Gr.
November 18: Social evening.
December 2: "Early life assurance in Britain" by Hudson Stowe '22; "Newton" by George Tuck '23.
December 16: "The notion of correspondence in mathematics" by Professor Samuel Beatty.
January 6, 1921: "My first impressions of the M. P. course" by Dorothy Gavin '22; "Laplace" by Harley Dewey '22.
January 20: Skating party.
February 3: "The life and significance of Galois" by Professor A. T. DeLury. Open meeting.
February 17: Debate between First and Second Years—"Resolved that individual initiative is more apt to bring success than is state control in the development of industrial enterprises."
March 3: "A magnetic survey" by Joseph Pearce, Gr.; "The history of mathematics" by Ewen Armstrong '21.
March 17: "Examination howlers" by Professor J. Satterly. Election of officers. The closing meeting.

(From a printed program, supplemented by Professor Beatty.)

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. P. MANNING.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known text-books, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

2951. Proposed by B. F. FINKEL, Drury College.

What is the average linear velocity of a point on the periphery of a locomotive driving wheel 2 feet in diameter and making 100 revolutions per minute?

2952. Proposed by R. M. MATHEWS, Wesleyan University.

Problem 2900 (*1921*, 277) suggests the following: Derive formulas for the products:

$$\cos\theta \cos\frac{1}{2}\theta \cos\frac{1}{2^2}\theta \cdots \cos\frac{1}{2^n}\theta,$$

$$\sin\theta \sin\frac{1}{2}\theta \sin\frac{1}{2^2}\theta \cdots \sin\frac{1}{2^n}\theta,$$

and

$$\tan\theta \tan\frac{1}{2}\theta \tan\frac{1}{2^2}\theta \cdots \tan\frac{1}{2^n}\theta.$$

2953. Proposed by C. F. GUMMER, Queen's University.

An algebraic equation being known to have exactly r real roots, all simple, is it possible to find an equation of degree $r - 1$ whose roots separate those of the given equation, and whose coefficients are rational functions of the coefficients of the given equation?

2954. Proposed by C. N. MILLS, Heidelberg University.

A machine-gun is placed on an armored train which is moving with a velocity of v feet per second along a straight horizontal track. The muzzle velocity of the bullets is v feet per second. Find the greatest range, (1) in front of the train and (2) behind the train.

2955. Proposed by the late L. G. WELD.

Find the proportions of an anchor ring such that its section by a plane parallel to its axis and tangent to its inner circle (circle of the gorge) shall be a lemniscate.

2956. Proposed by W. D. LAMBERT, U. S. Coast and Geodetic Survey.

A particle is constrained to move on the outer surface of a sphere whose radius equals the mean radius of the earth and which is rotating with the same angular velocity, ω, as the earth. The particle is acted upon by a force equal to $m\omega^2 \sin\phi \cos\phi$, which is directed to the nearer pole, m being the mass of the particle and ϕ the latitude. This force balances the equatorward tendency of a particle initially at rest relative to the surface; on the earth its place is supplied by the slight departure from sphericity. Show that for small velocities relative to the rotating surface the particle when acted upon by no forces except the one mentioned and by the constraint, will describe a curve differing slightly from a small circle on the sphere.

Discuss the form of the curve when the velocity relative to the surface is increased. Would it be possible to lay out a railroad track from New York to Chicago such that its departure from the great circle connecting those points would counterbalance the pressure on the rails due to the earth's rotation?

2957. Proposed by J. L. WALSH, Harvard University.

The envelope of the circles of curvature of a curve is, in part at least, the curve itself. What further curves, if any, are parts of this envelope?

2958. Proposed by R. P. BAKER, University of Iowa.

Over a frictionless pulley a weightless cord sustains at one end a mass M, while the other end is wound on the axle of a wheel of mass M and moment of inertia N. At the zero of time the wheel revolves with angular velocity ω and tends to wind up the cord. Describe the motion neglecting friction.

SOLUTIONS

2813 [1920, 81]. Proposed by PAUL CAPRON, U. S. Naval Academy.

An ellipse having the major-axis $2a$ and the eccentricity ϵ, is revolved first about its major axis, forming a prolate spheroid, then about its minor axis forming an oblate spheroid. Show that the surfaces of these spheroids are, respectively,

$$2\pi a^2(1/\epsilon)(\sqrt{1-\epsilon^2}\sin^{-1}\epsilon + 1)$$

and

$$2\pi a^2\left[2 + (1/\epsilon)(1-\epsilon^2)\log\left(\frac{1+\epsilon}{1-\epsilon}\right)\right].$$

SOLUTION BY H. S. UHLER, Yale University.

Case I. Prolate spheroid.

Let the equation of the ellipse be

$$b^2x^2 + a^2y^2 - a^2b^2 = 0. \tag{1}$$

We may take as element of surface the lateral area of the frustum of a right circular cone having the following specifications: planes of bases normal to the x-axis and at distances x and $x + dx$ from the origin, and generatrix of lateral surface tangent to the revolving ellipse at the point: (x, y). Slant height $\equiv dt$.

Then

$$dt = dx\sqrt{1 + (dy/dx)^2}.$$

From (1)

$$\frac{dy}{dx} = -\frac{b^2x}{a^2y};$$

hence

$$dt = \frac{bdx}{a^2y}\sqrt{a^4 - (a^2-b^2)x^2}.$$

Accordingly, the element of surface, $d\sigma$, is given by

$$d\sigma = 2\pi y dt = \frac{2\pi b dx}{a^2}\sqrt{a^4 - (a^2-b^2)x^2},$$

and the required area of the prolate spheroid,

$$A_1 = \frac{4\pi b}{a^2}\int_0^a dx\sqrt{a^4 - (a^2-b^2)x^2}. \tag{2}$$

By making use of the standard formula,

$$\int d\varphi\sqrt{n^2 - \varphi^2} = \frac{1}{2}\varphi\sqrt{n^2-\varphi^2} + \frac{1}{2}n^2\sin^{-1}\frac{\varphi}{n},$$

it is merely a matter of simple reductions to change equation (2) to the following form,

$$A_1 = 2\pi b\left[b + \frac{a^2}{\sqrt{a^2-b^2}}\sin^{-1}\left(\frac{\sqrt{a^2-b^2}}{a}\right)\right]. \tag{3}$$

Now, by the definition of e, $b^2 = a^2(1-e^2)$; so that, by elimination of b, equation (3) may be expressed as the following function of a and e:

$$A_1 = 2\pi a^2(e^{-1}\sqrt{1-e^2}\sin^{-1}e + 1 - e^2). \tag{4}$$

Case II. Oblate spheroid.

Proceeding as above, it will be found that

$$A_2 = \frac{4\pi a}{b^2}\int_0^b dy\sqrt{(a^2-b^2)y^2 + b^4}.$$

The following well-known relation

$$\int d\varphi\sqrt{\varphi^2+n^2} = \tfrac{1}{2}\varphi\sqrt{\varphi^2+n^2} + \tfrac{1}{2}n^2\log(\varphi+\sqrt{\varphi^2+n^2})$$

facilitates the reduction of the preceding integral to

$$A_2 = 2\pi a\left[a + \frac{b^2}{\sqrt{a^2-b^2}}\log\left(\frac{a+\sqrt{a^2-b^2}}{b}\right)\right],$$

or

$$A_2 = 2\pi a^2\left[1 + \frac{1}{2}e^{-1}(1-e^2)\log\left(\frac{1+e}{1-e}\right)\right]. \tag{5}$$

Remarks. Both of the proposed expressions are incorrect. When $e = 1$ ($b = 0$) the given formula for the prolate spheroid leads to $2\pi a^2$ instead of zero. Again, when $e = 0$ ($b = a$) the series

$$\frac{1}{2}e^{-1}\log\left(\frac{1+e}{1-e}\right) = 1 + \frac{1}{3}e^2 + \frac{1}{5}e^4 + \cdots$$

shows that the given expression for the oblate spheroid reduces to $8\pi a^2$ instead of $4\pi a^2$ (sphere). Formulas (4) and (5) are equivalent to those given in Williamson's *Integral Calculus*, page 258.

Note. Contributors named below observed that the results as corrected are given in Lamb's *Infinitesimal Calculus*, 1902, page 273 and in Czuber's *Integralrechnung*, page 284. It is also given in *The Encyclopaedia Britannica*, ninth edition, volume 13, page 55.—EDITORS.

Also solved by NORMAN ANNING, J. A. BULLARD, NATHAN DEUTSCHMAN, A. FAYDER, L. G. FEMAN, MAURICE KRAUT, BENJAMIN LEVINE, I. H. MARANTZ, MOSES NISSENBAUM, H. L. OLSON, ARTHUR PELLETIER, H. W. REDDICK, H. A. RHODES, J. L. RILEY, D. H. RICHERT, J. B. REYNOLDS, and T. R. THOMSON.

2815 [1920, 134]. Proposed by the late L. G. WELD.

A right circular cone is laid upon an inclined plane so that its element of contact makes a given angle with the slant line of the plane. Assuming that there is no slipping and that the rolling friction is negligible, find the time of oscillation of the cone.

I. SOLUTION BY J. B. REYNOLDS, Lehigh University.

The center of gravity of the cone moves in a circle of radius $\frac{3}{4}h\cos\alpha$ at a perpendicular distance $\frac{3}{4}h\sin\alpha$ from the plane, h being the height of the cone and 2α its vertical angle. When the radius of this circle makes an angle θ with the line of slope of the plane, the center of gravity of the cone has been raised a height $\frac{3}{4}h\cos\alpha(1-\cos\theta)\sin\beta$ above its lowest position, β being the angle of slope of the plane. Then the potential energy, *P.E.*, is $\frac{3}{4}mgh\cos\alpha\sin\beta(1-\cos\theta)$, m being the mass of the cone.

Since the line of contact is the instantaneous axis of rotation, the kinetic energy, *K.E.*, of the cone is $\frac{1}{2}I\dot{\varphi}^2$, I being the moment of inertia with respect to the instantaneous axis or an element of the cone and $\dot{\varphi}$ the angular velocity about this axis.

Now the moment of inertia of a thin disc of radius a and thickness t with respect to a line through its center making an angle α with a perpendicular to its plane is

$$t\int_0^{2\pi}\int_0^{a} r^2(\cos^2\theta\cos^2\alpha + \sin^2\theta)rdrd\theta = \frac{\pi a^4}{4}(1+\cos^2\alpha)t$$

and for the moment of inertia of the disc about a parallel line at a distance $a\cos\alpha$, we have

$$\frac{\pi a^4}{4}(1+\cos^2\alpha)t + \pi a^2 t(a\cos\alpha)^2 = \frac{\pi a^4}{4}(1+5\cos^2\alpha)t.$$

So that, since the radius of any circular differential disc at distance x from the vertex of the cone

is $x \tan \alpha$, we have

$$I = \int_0^h \frac{\pi}{4}(1 + 5\cos^2\alpha)\tan^4\alpha\, x^4 dx = \frac{\pi}{20}h^5(1 + 5\cos^2\alpha)\tan^4\alpha$$

$$= \tfrac{3}{20}mh^2\tan^2\alpha(1 + 5\cos^2\alpha),$$

since

$$m = \tfrac{1}{3}\pi h^3 \tan^2\alpha.$$

By the principle of the conservation of energy

$$K.E. + P.E. = \text{constant}$$

or

$$\tfrac{1}{2}\tfrac{3}{20}mh^2\tan^2\alpha(1 + 5\cos^2\alpha)\dot{\varphi}^2 + \tfrac{3}{4}mgh\cos\alpha\sin\beta(1 - \cos\theta) = \text{constant};$$

or since $\dot{\varphi} = \dot{\theta}\cot\alpha$, we get

$$\tfrac{1}{10}h^2(1 + 5\cos^2\alpha)\dot{\theta}^2 + gh\cos\alpha\sin\beta(1 - \cos\theta) = \text{constant}.$$

Differentiating this equation with respect to t and cancelling out $\dot{\theta}$ we get,

$$\ddot{\theta} = -\frac{5g\cos\alpha\sin\beta}{h(1 + 5\cos^2\alpha)}\sin\theta,$$

from which we have, if θ is small, for the time T for an oscillation

$$T = 2\pi\sqrt{\frac{h(1 + 5\cos^2\alpha)}{5g\cos\alpha\sin\beta}}.$$

If the oscillations are not small and γ is the given angle that the element of contact initially makes with the line of greatest slope

$$T = 2\pi\sqrt{\frac{g(1 + 5\cos^2\alpha)}{5g\cos\alpha\sin\beta}}\left\{1 + \frac{1}{4}\sin^2\frac{\gamma}{2} + \frac{9}{64}\sin^4\frac{\gamma}{2} + \cdots + \left[\frac{n\lfloor 2n-1}{2^{2n-1}(\lfloor n)^2}\right]^2\sin^{2n}\frac{\gamma}{2} + \cdots\right\}.$$

II. Solution by William Hoover, Columbus, Ohio.

Assume the vertex of the cone as fixed in the inclined plane; then we have the motion of a solid of revolution with one point fixed; and the motion is represented by Euler's equations, employing the usual notation, in the form

$$\begin{aligned} A\dot{\omega}_1 + (C - A)\omega_2\omega_3 &= L, \\ A\dot{\omega}_2 - (C - A)\omega_3\omega_1 &= M, \\ C\dot{\omega}_3 \qquad\qquad\quad &= N, \end{aligned} \tag{1}$$

the two equal moments of inertia about axes through the vertex of the cone and perpendicular to the axis of the cone and to each other being given by A, and the moment of inertia about the axis of the cone by C.

Also take the line of contact of the cone and plane as an instantaneous axis, and let its direction angles be λ, μ, ν, and the vertical angle of the cone $= 2\alpha$, φ = the angle through which the cone rotates after any time t from the beginning of motion; so that, the x-axis being initially in the inclined plane, φ also gives the rotary motion of the x-axis; and ω = the instantaneous angular velocity about the instantaneous axis.

Let ρ = the distance of any point in the instantaneous axis from the origin; then as in solid analytic geometry,

$$\cos\lambda = \frac{x}{\rho} = \sin\alpha\sin\varphi, \qquad \cos\mu = \frac{y}{\rho} = \sin\alpha\cos\varphi, \qquad \cos\nu = \cos\alpha; \tag{2}$$

and, by the theory of rotary motion,

$$\omega_1 = \omega\sin\alpha\sin\varphi, \qquad \omega_2 = \omega\sin\alpha\cos\varphi, \qquad \omega_3 = \omega\cos\alpha. \tag{3}$$

We have

$$\omega = \dot{\varphi}. \tag{4}$$

If m = the mass of the cone, and h = its altitude,

$$A = \tfrac{3}{20}mh^2(4 + \tan^2\alpha), \qquad C = \tfrac{3}{10}mh^2\tan^2\alpha. \tag{5}$$

Let $\dot{\theta}$ = the angular motion of the instantaneous axis on the plane; this is the angular motion about the line (n) which is normal to the inclined plane and passes through the vertex of the cone. A plane perpendicular to the inclined plane includes this normal, the instantaneous axis (a_i), the axis of the cone, and a perpendicular (p) to the axis of the cone. The angle between n and p is α and between a_i and p is $\pi/2 + \alpha$. The components of $\dot{\theta}$ and ω about p are $\dot{\theta}\cos\alpha$ and $\omega\cos(\pi/2 + \alpha) = -\omega\sin\alpha$, but these denote the same motion; then

$$\dot{\theta}\cos\alpha = -\omega\sin\alpha \qquad \text{or} \qquad \omega = -\dot{\theta}\cot\alpha. \tag{6}$$

Now the moment of the external forces about the instantaneous axis

$$= L\cos\lambda + M\cos\mu + N\cos\nu; \tag{7}$$

also

$$= \tfrac{3}{4}mgh\sin\alpha\sin\beta\sin\theta \tag{8}$$

β being the inclination of the plane.

Substituting (4) in (3) and differentiating, we have

$$\dot{\omega}_1 = \sin\alpha(\cos\varphi\cdot\dot{\varphi}^2 + \sin\varphi\cdot\ddot{\varphi}), \quad \dot{\omega}_2 = \sin\alpha(-\sin\varphi\cdot\dot{\varphi}^2 + \cos\varphi\cdot\ddot{\varphi}), \quad \text{and} \quad \dot{\omega}_3 = \cos\alpha\cdot\dot{\varphi}, \tag{9}$$

Now put the values given by (3), (5), (9) in (1), and we have the values of L, M, N; and using these, with (2), (7) and (8), we have the single equation of motion

$$\frac{3m}{20}(6 + \tan^2\alpha)h^2\sin^2\alpha\cdot\ddot{\varphi} = \frac{3m}{4}gh\sin\alpha\sin\beta\sin\theta. \tag{10}$$

Now from (4) and (6),

$$\ddot{\theta} = -\frac{g}{h}\frac{5\sin\beta\cos\alpha}{1 + 5\cos^2\alpha}\sin\theta. \tag{11}$$

If θ be so small that θ may displace $\sin\theta$, we have the required time of oscillation

$$T = 2\pi\sqrt{\frac{1 + 5\cos^2\alpha}{5\sin\beta\cos\alpha}\cdot\frac{h}{g}}.$$

Also solved by F. L. WILMER, and discussed by C. H. ECKART.

2820 [1920, 134]. Proposed by C. B. HALDEMAN, Ross, Ohio.

Given one angle and the radii of the inscribed and circumscribed circles, to construct the triangle geometrically.

SOLUTION BY A. V. RICHARDSON, Bishop's College, Lennoxville, Quebec.

With the usual notation, A, R, r are given. Draw a circle of radius R, and let KB be any diameter. (The reader is requested to draw the figure). Make the angle, $BKC = A$. Draw XY parallel to BC and at a distance r from it on the same side as K. Bisect the arc BC at V, and let the circle, center V, radius VC, cut XY at I and I'.

These points will be the incenters for the two (symmetrical) solutions. Let VI meet the circle again in A. Then ABC is the required triangle. To show this, it is only necessary to prove that the straight line CI bisects the angle C. Since $VI = VC$, $\angle VIC = \angle VCI$. But $\angle VIC = (A/2) + \angle ICA$ and $\angle VCI = (A/2) + \angle BCI$. Hence $\angle ICA = \angle BCI$ and I is the intersection of two bisectors of the angles of the triangle ABC.

Also solved by L. C. MATHEWSON, H. L. OLSON, ARTHUR PELLETIER, J. B. REYNOLDS, JOSEPH ROSENBAUM, C. N. SCHMALL, and the Proposer.

2829 [1920, 226]. Proposed by E. S. PALMER, New Haven, Conn.

Given a set of arbitrary pairs of positive integers (a_p, b_p), $(p = 1, 2, \cdots, n)$: (a) Is it always possible to find a set of positive integers k_p, $(p = 1, 2, \cdots, n)$ such that

$$k_p a_p + k_p b_p > \sum_{r=1}^{r=n} k_r a_r, \qquad (p = 1, 2, 3, \cdots, n).$$

(b) If or when possible, show how to find k_p.

Solution by Albert A. Bennett, University of Texas.

The system is homogeneous, so that if one solution exists, an infinite number exist. For convenience write c_p for $a_p + b_p$ and write in place of the inequalities the following equalities:

$$k_p c_p = \sum_{r=1}^{n} k_r a_r + e_p, \qquad e_p > 0, \qquad (p = 1, 2, 3, \cdots, n).$$

Solving for k_p, one has, provided that the denominators do not vanish,

$$k_p = \frac{e_p}{c_p} + \frac{1}{c_p} \frac{\sum_{r=1}^{n} \frac{e_r a_r}{c_r}}{1 - \sum_{r=1}^{n} \frac{a_r}{c_r}},$$

as may be verified at once by substitution.

Three cases occur.

1. $1 - \sum_{r=1}^{n} \frac{a_r}{c_r} > 0$. In this case, solutions are obtained readily by inserting arbitrary e's restricted only so that the k's shall be integers.

2. $1 - \sum_{r=1}^{n} \frac{a_r}{c_r} = 0$. There are no solutions in this case.

3. $1 - \sum_{r=1}^{n} \frac{a_r}{c_r} < 0$. In this case, also, there are no solutions. For if there were solutions, these would be in the form given above, and the expression, $\sum_{p=1}^{n} k_p a_p$, obtained by multiplying each solution, k_p by a_p and adding together would be

$$\sum_{p=1}^{n} k_p a_p = \sum_{p=1}^{n} \frac{e_p a_p}{c_p} + \sum_{p=1}^{n} \frac{a_p}{c_p} \frac{\sum_{r=1}^{n} \frac{e_r a_r}{c_r}}{1 - \sum_{r=1}^{n} \frac{a_r}{c_r}},$$

$$= \sum_{r=1}^{n} \frac{e_r a_r}{c_r} \frac{1}{1 - \sum_{r=1}^{n} \frac{a_r}{c_r}}.$$

Since the right-hand member would be negative under the hypothesis of this case, there could be no set of positive members, k_p, $p = 1, 2, \cdots, n$, of the form required for a solution. (The writer is indebted to Prof. H. P. Manning for the treatment here given of this third case.)

2833 [1920, 227]. Proposed by W. H. Echols, University of Virginia.

In *Engineering*, London, September 28, 1917, appeared the following equations concerning the stability of ships; they are employed by the naval constructors in the Norfolk (Virginia) Navy Yard, and they are of importance:

$$\tan^3 \theta_0 + \frac{2m}{\rho} \tan \theta_0 - \frac{2x_0}{\rho} = 0, \qquad \tan^3 (\theta_0 + \theta_1) + \frac{2m}{\rho} \tan (\theta_0 + \theta_1) - \frac{2x_0}{\rho} - \frac{2x}{\rho} = 0,$$

$$\tan^3 (\theta_0 - \theta_2) + \frac{2m}{\rho} \tan (\theta_0 - \theta_2) - \frac{2x_0}{\rho} + \frac{2x}{\rho} = 0.$$

The required unknowns are θ_0, x_0 and m. The constants have values as follows: θ_1 and θ_2 are positive angles ranging from 15′ to 10°, x is positive and less than 5, and ρ is positive with a considerable range of values. A rapid solution involving small labor is desired, determining x_0 within the same limits as given for x.

Solution by the Proposer.

It may be of interest to note that in the practical application of the problem, ρ is the metacentric radius when the vessel is upright, x is the shift of the center of gravity of the ship when a weight w is placed on the side at a distance d from the center line and is equal to wd/W, where W is the weight of displaced water. In a typical ship of 4000 tons, w equal to 2.5 tons, $\rho = 16$ ft.,

$x = 0.025$ ft., the constant $2x/\rho$ is 0.003125. The required m is the metacentric height, or the distance from the center of gravity of the ship to the metacenter when the vessel is upright. The "angle of heel," θ, is the angle through which the ship turns when the weight w is shifted transversely, and is small in the case of testing an ordinarily well designed ship. Under these circumstances the following solution appears to be valid.

Let

$$X = 2m/\rho, \qquad Y = 2x_0/\rho, \qquad a = 2x/\rho.$$

Then eliminating X and Y there results

$$\begin{vmatrix} \tan^3\theta_0, & \tan\theta_0, & 1 \\ \tan^3(\theta_0+\theta_1)-a, & \tan(\theta_0+\theta_1), & 1 \\ \tan^3(\theta_0-\theta_2)+a, & \tan(\theta_0-\theta_2), & 1 \end{vmatrix} = 0. \tag{1}$$

On expanding the determinant, (1) becomes

$$[\tan(\theta_0+\theta_1)-\tan\theta_0][\tan(\theta_0-\theta_2)-\tan\theta_0][\tan(\theta_0+\theta_1)-\tan(\theta_0-\theta_2)][\tan(\theta_0+\theta_1)$$
$$+\tan(\theta_0-\theta_2)+\tan\theta_0]-a[\tan(\theta_0+\theta_1)+\tan(\theta_0-\theta_2)-2\tan\theta_0]=0. \tag{2}$$

In (2) put $z = \tan\theta_0$, $p = \tan\theta_1$, $q = \tan\theta_2$. On simplifying, the equation becomes

$$pq(p+q)\frac{(1+z^2)^2}{(1-pz)^2(1+qz)^2}\{pqz^3+2(p-q)z^2-(3+2pq)z-(p-q)\} - a\{2pqz+(p-q)\}=0, \tag{3}$$

a factor

$$\frac{1+z^2}{(1-pz)(1+qz)} \equiv K \tag{4}$$

being omitted since it does not involve the solution. Interchanging p and q in (3) merely changes the sign of z, therefore we may assume $p \geqq q$.

The angle of heel, θ_0, of the unweighted ship, due to faulty construction by which the material is not so distributed as to keep the center of gravity in a vertical plane of symmetry, is in general small. The expression represented by K in (4) is nearly unity under the conditions.

The roots of equation (3) are the abscissas of the points of intersection of the straight line

$$y = 2apqz + a(p-q), \tag{5}$$

and the curve

$$y = pq(p+q)K^2\{pqz^3+2(p-q)z^2-(3+2pq)z-(p-q)\}. \tag{6}$$

The curve (6) cuts the z-axis in the same points as does the cubic

$$y = pq(p+q)\{pqz^3+2(p-q)z^2-(3+2pq)z-(p-q)\}, \tag{7}$$

and it has vertical asymptotes $z = 1/p$ and $z = -1/q$. The ordinates of (7) at $z = -1/q$ and $z = 1/p$ are respectively

$$+pq(p+q)\left(\frac{p}{q^2}+\frac{1}{q}+p+q\right) \quad \text{and} \quad -pq(p+q)\left(\frac{q}{p^2}+\frac{1}{p}+p+q\right).$$

The cubic has three real roots γ, α, β in the respective intervals

$$-\infty, \qquad -1/q, \qquad +1/p, \qquad +\infty,$$

and cuts the y-axis at $y = -pq(p^2-q^2)$. It has a concavo-convex inflection at

$$z = -\frac{2}{3}\frac{p-q}{pq},$$

the ordinate there being positive. The middle root α lies between the origin and the z-intercept of the tangent at $z = 0$, which is

$$z = -\frac{p-q}{3+2pq}.$$

The straight line (5) cuts the z-axis midway between the asymptotes at a point

$$A = -(p-q)/2pq.$$

The slope of (5) is positive and this line (as we shall show) cuts (6) in only one point between the asymptotes. The abscissa of this point is the root we seek.

Represent by Q the cubic cofactor of K^2 in (6). Take the logarithm of both sides of that equation and differentiate. Then

$$\frac{1}{2y}\frac{dy}{dz} = \frac{2z}{1+z^2} + \frac{(p-q)+2pqz}{(1-pz)(1+qz)} + \frac{1}{2Q}\frac{dQ}{dz}. \tag{8}$$

The ordinate y is positive between A and α, so also is Q. The first term on the right is negative when z is negative. The derivative DQ is negative from $z = 0$ to its negative root which is easily seen to be less than $-1/q$. The inflection of $y = Q$ occurs at $(4/3)A$.

The numerator obtained by adding the first two terms on the right is

$$(p-q) + 2(1+pq)z - (p-q)z^2.$$

This is clearly negative for $z = -(p-q)/[2(1+pq)]$ and all smaller values, and therefore from A to $-\frac{1}{2}(p-q)$, inclusive. Hence the derivative Dy is negative throughout this interval.

Consider the interval from $-\frac{1}{2}(p-q)$ to α. The derivative DQ decreases in absolute value as x varies from $-\frac{1}{2}(p-q)$ to 0. The greatest value of Q in the interval from $-\frac{1}{2}(p-q)$ to α is at $-\frac{1}{2}(p-q)$. The numerical value of the last term in (8) is greater than that of DQ taken at $z = 0$ divided by twice that of Q at $-\frac{1}{2}(p-q)$, or

$$\frac{3+2pq}{(p-q)\{1+2pq+(1-\frac{1}{4}pq)(p-q)^2\}},$$

which is greater than

$$\frac{3}{p(1+2p^2)}.$$

The second term on the right in (8) is 0 at A and increases (the numerator increasing and the denominator decreasing) through positive values from $z = A$ to $z = 0$ when it is greatest and equal to $p - q$. Therefore this term is less than p throughout the interval from $-\frac{1}{2}(p-q)$ to α. The sum of the last two terms on the right in (8) is certainly negative for values of p which make $p^2(1+2p^2) < 3$, which is true if $p < 1$, or $\theta_1 < 45°$.

These results show that y in (6) is a one-valued continuously decreasing function of z throughout the interval A to α and therefore there is only one root of (3) in this interval.

The conditions of the problem are such that the required root of (3) is small; a first and practically sufficient approximation

$$z_1 = -\frac{p-q}{pq}\,\frac{a+pq(p+q)}{2a+(p+q)(3+2pq)}, \tag{9}$$

is obtained by making K provisionally equal to 1 and neglecting terms in Q containing powers of z above the first. A new approximation can be obtained by putting z_1 for z in K and in z^3 and z^2 in Q, and solving again for z.

It is of interest to note that the original equations can be written

$$Y = \alpha X + \alpha^3, \qquad Y + a = \beta X + \beta^3, \qquad Y - a = \gamma X + \gamma^3,$$

regarding α, β, γ as variable parameters. These represent three straight lines which envelope, respectively, the semi-cubic parabolas

$$27Y^2 + 4X^3 = 0, \qquad 27(Y+a)^2 + 4X^3 = 0, \qquad 27(Y-a)^2 + 4X^3 = 0.$$

These straight lines make given angles with each other. The second makes $+\theta_1$ with the first and the third makes $-\theta_2$ with the first.

A graphical solution of the equations consists in fitting a figure composed of three straight lines, meeting in a point and making given angles with each other, so that each is tangent to the corresponding semi-cubic parabola. The intersection of these three tangents has the required coördinates X, Y. The slope of the tangent to the first parabola is the required $\tan\theta_0$.

An actual navy yard experiment gave the data

$$\theta_1 = 4° 49' 14'', \qquad \rho = 2.63, \qquad \theta_2 = 4° 39' 36'', \qquad x = 0.0568.$$

Application of the computations indicated above gives $p = 0.0843$, $q = 0.0815$, $a = 0.043$?, $\tan\theta_0 = -0.03124$, $\theta_0 = -(1° 47' 21'')$. $X = 0.511$, $Y = -0.016$, $m = 1.344$, $x_0 = -0.0$?.

2835 [1920, 273]. Proposed by J. L. RILEY, Stephenville, Texas.

If x, y, z, u are finite, and not all zero, and satisfy the equations

$$x = by + cz + du, \quad y = ax + cz + du, \quad z = ax + by + du, \quad u = ax + by + cz,$$

and if none of the quantities a, b, c, d have the value -1, then will

$$\frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c} + \frac{d}{1+d} = 1.$$

SOLUTION BY H. L. OLSON, University of Michigan.

The necessary and sufficient condition that these equations shall have a set of solutions not all zero is that the determinant of the coefficients,

$$\begin{vmatrix} -1, & b, & c, & d \\ a, & -1, & c, & d \\ a, & b, & -1, & d \\ a, & b, & c, & -1 \end{vmatrix} = 0.$$

If we expand this determinant and divide by $(a+1)(b+1)(c+1)(d+1)$, which is not zero by hypothesis, we find that the condition can be written

$$\frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c} + \frac{d}{1+d} = 1.$$

Also solved by P. J. DA CUNHA, ARTHUR PELLETIER, A. V. RICHARDSON, C. H. RICHARDSON, H. S. UHLER, and C. C. WYLIE.

2857 [1920, 377]. Proposed by the late L. G. WELD.

A savings bank offers to pay 3% interest on deposits, the said interest to be continuously compounded, *i.e.*, compounded at infinitesimal intervals of time. What would be the amount of $1.00 for one year?

SOLUTION BY E. J. OGLESBY, Washington Square College, New York University.

The amount of one dollar at the percentage, denoted by the fraction r, compounded n times per year is $(1 + r/n)^n = [(1 + r/n)^{n/r}]^r$. Hence

$$A = \lim_{n \doteq \infty} \left[\left(1 + \frac{r}{n} \right)^{n/r} \right]^r = e^r = e^{.03} = 1.03045.$$

Also solved by H. C. BRADLEY, H. N. CARLETON, ELMER LATSHAW, ARTHUR PELLETIER, W. T. STONE, and A. L. WECHSLER.

2858 [1920, 428]. Proposed by C. P. SOUSLEY, Pennsylvania State College.

A boy can split wood as fast as his father can saw, and the father can split twice as fast as the son can saw. How should the money received for their labor be divided?

SOLUTION[1] BY C. A. BARNHART, University of New Mexico.

Let x = the number of cords of wood that the boy saws in 1 hour, and kx = the number of cords of wood that the boy splits in 1 hour. Then, $2x$ = the number of cords of wood that the father splits in 1 hour, and kx = the number of cords of wood that the father saws in 1 hour.

Then, $\frac{1}{x} + \frac{1}{kx} = \frac{k+1}{kx}$ = the number of hours in which the boy will saw and split 1 cord of wood, and $\frac{1}{2x} + \frac{1}{kx} = \frac{k+2}{2kx}$ = the number of hours in which the father will saw and split 1 cord of wood.

[1] The question is rather vague, but probably this solution makes the most natural interpretation of it: to use the time it takes each to split and saw a cord of wood as a basis for comparison of their work.—EDITORS.

Assuming that for each dollar received by the son, the father will receive z dollars, and noting that the money received will vary inversely as the number of hours required to saw and split 1 cord of wood, we have

$$z = \frac{k+1}{kx} \Big/ \frac{k+2}{2kx} = 1 + \frac{k}{k+2}.$$

From which it is evident that the money received by the father depends directly upon k. Since $\lim_{k \doteq 0} z = 1$ and $\lim_{k \doteq \infty} z = 2$, then for $k > 0$, we have $1 < z < 2$.

Also solved by C. E. BARDSLEY, H. C. BRADLEY, H. N. CARLETON, MICHAEL GOLDBERG, and DANIEL KRETH.

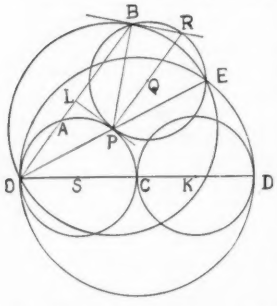
2861 [1920, 428]. Proposed by B. F. FINKEL, Drury College.

Obtain by plane geometry, *i.e.*, without the use of calculus, a construction for finding points on the envelope of a system of circles whose diameters are chords of a fixed circle passing through a given point on it. Also determine geometrically the nature of the locus.

I. SOLUTION BY AUGUSTUS BOGARD, College of St. Theresa, Winona, Minn.

The problem of finding the envelope of a system of curves is that of finding a curve that is at every point tangent to some curve of the system. That is, at any point on the envelope the envelope and some curve of the system have a common tangent, and they lie on the same side of it.

Let O be the fixed point, C the center of the given circle, and OCD the diameter from O, and let OE be any other chord from O, making with OD the angle θ. OE is then a diameter and P, its mid-point, is the center of one of the circles of the given system. At P, on the further side of OE, construct an angle $EPB = 2\theta$, the side of the angle cutting the circle OE at B. Then B is a point of the envelope required.



Proof. Draw the circles having OC and CD as diameters, and the circle PEB. Let Q be the center and PQR the diameter from P of PEB. PQR bisects the angle EPB and this circle is tangent at P to the circle OC. Further, chord PB = chord PO and so diameter PR = diameter OC = diameter CD. Hence the circle whose center is at Q may be thought of as rolling without slipping on the circle OC, and P is the center of the curvature at B of the curve described by this point. It follows that the line BR is tangent to the curve at B. It is also tangent at B to the circle OE, the circle of the given system, passing through B. Hence B is a point on the envelope. This analysis shows that the curve is described in the same manner as the cardioid, and is therefore a cardioid. Its polar equation is easily derived as follows: Considering O as the pole, OD as the initial line, the angle $DOB = \phi$ as the vectorial angle of B, and OB its radius vector, we have $OB = OE \cos\theta = OD \cos^2\theta = a(\cos 2\theta + 1) = a(1 + \cos\phi)$, where $a = OC$. That is, $\rho = a(1 + \cos\phi)$, which is the equation of a cardioid.

II. SOLUTION BY OTTO DUNKEL, Washington University.

Using the notation in A. Bogard's solution, the middle point P of the chord OE of the fixed circle, of radius OC, lies on the circle OC. It may be shown easily that the tangent at P to the circle OC is parallel to the tangent to the given circle at E. Hence the construction given for the point B in the MONTHLY (*1921*, 182) is equivalent to the following: Draw the tangent at E to the given fixed circle, then the point B is the foot of the perpendicular from O to the tangent. Let A be the point in which OB cuts the circle OC, then $AB = CE = a$. Hence the point B is found by the usual method for constructing a cardioid, *i.e.*, by drawing chords OA to the circle OC and prolonging each a length $AB = OC = a$.

III. Solution, Historical Notes, and Remarks by R. C. Archibald, Brown University.

In an article by T. de St-Laurent, *Annales de Mathématiques Pures et Appliquées*, July, 1826, vol. 17, page 15, it is found that the cardioid is the catacaustic of a reflecting circle with respect to a luminous point on its circumference. In a footnote to this result Gergonne remarked that the catacaustic is the envelope of the space traversed by a moving circle of variable radius, constantly having its centre on the circumference of a circle concentric with the reflecting circle [and of one third the radius], and passing through a fixed point of this circumference.

But the result is only a very special case of a well-known theorem by Quetelet:[1] The envelope of the circles passing through a fixed point, and whose centers lie on a given curve, C, is the pedal with respect to the fixed point of a curve similar to C but of double its linear dimensions.

Colin Maclaurin, in a memoir[2] written in 1718, at twenty years of age, was not only the first to show that the cardioid is the first positive pedal of a circle, with respect to a point on its circumference, but also the first to find the equations of the positive and negative pedals of the cardioid with respect to its cusp.

That the envelope of the circles in B. F. Finkel's problem is a cardioid was shown by B. Price who gave an analytic proof[3] in 1852. A geometrical proof of this result is readily derived by inversion[4] of the theorem: If the vertex of a right angle move along a straight line and one side pass through a fixed point O, the envelope of the other side will be a parabola with focus at O.

Conversely, if a series of circles through the cusp of a cardioid are tangent to it, their cuspidal diameters will be chords of the cardioid's axial circle (Stubbs,[5] 1843).

It is interesting to note the relation of Quetelet's theorem to Otto Dunkel's solution above, and to his paper "The relation of caustics to certain envelopes" (1921, 182–183).

Also solved by Joseph Rosenbaum, A. V. Richardson, and F. L. Wilmer.

2864 [1920, 482]. Proposed by C. B. HALDEMAN, Ross, O.

If S is a side of a regular undecagon inscribed in a circle of radius unity, show that

$$S^5 - (S^4 - 3S^2 - 1)\sqrt{11} - 11S = 0.$$

Solution by A. M. Harding, University of Arkansas.

It is evident that $S = 2 \sin \pi/11$. Hence our problem is to find an equation one of whose roots is $2 \sin \pi/11$. The complex eleventh roots of unity, which occur in conjugate pairs, viz.

$$\cos \frac{2k\pi}{11} \pm i \sin \frac{2k\pi}{11}, \qquad k = 1, 2, 3, 4, 5.$$

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[1] *Nouveaux Mémoires de l'Académie de Bruxelles*, vol. 3, 1826, p. 91 (memoir presented February 3, 1823); the theorem is stated as follows: La caustique par refléxion pour une courbe quelconque éclairée par un point brillant, est la developpée d'une autre courbe, laquelle a la propriété d'être l'envelope de tous les cercles qui ont leurs centres sur la courbe réfléchissante, et qui passent par le point brillant." Quetelet showed also that the "autre courbe" was a "secondary caustic," the locus determined by dropping a perpendicular, OL, from the luminous point, O, on the tangent to the reflecting curve, and producing this perpendicular an equal distance to B.

[2] C. Maclaurin, "Tractatus de curvarum constructione & mensura ubi plurimae series curvarum infinitae vel rectis mensurantur vel ad simpliciores curvas reducunter," *Philosophical Transactions of the Royal Society* of London, vol. 30, pp. 803–812; abridgment, vol. 6, 1809, pp. 356–362. The geometrical construction for points on the cardioid and its pedals is touched upon by Maclaurin in his *Treatise of Fluxions*, 1742, vol. 2.

[3] B. Price, *Infinitesimal Calculus*, vol. 1, 1852, p. 417. It was also proposed and solved in *Educational Times*, June and October, 1853, and June, 1856. There are many other discussions of the problem.

[4] C. Taylor, *An Introduction to the Ancient and Modern Geometry of Conics*, Cambridge, 1881, p. 357.

[5] J. W. Stubbs, "On the application of a new geometric method to the geometry of curves and surfaces," *Philosophical Magazine*, vol. 23, pp. 338–347.

satisfy the equation

$$\sum_{i=0}^{i=10} x^i = 0 \quad \text{or} \quad \sum_{i=1}^{i=5} \left(x^i + \frac{1}{x^i}\right) + 1 = 0. \tag{1}$$

By the substitution of y for $x + 1/x$ this equation in its second form may be reduced to[1]

$$y^5 + y^4 - 4y^3 - 3y^2 + 3y + 1 = 0. \tag{2}$$

By taking $k = 1$ it will be seen that one root of (2) is

$$y = 2\cos\frac{2\pi}{11} = 2 - 4\sin^2\frac{\pi}{11} = 2 - S^2, \quad \text{or} \quad -S^2 = z = y - 2.$$

Reducing by 2 the roots of (2) by synthetic division we obtain the equation for z

$$z^5 + 11z^4 + 44z^3 + 77z^2 + 55z + 11 = 0.$$

If we now substitute $-S^2$ for z and group the terms of the resulting equation we find

$$S^{10} - 22S^6 + 121S^2 = 11(S^8 - 6S^6 + 7S^4 + 6S^2 + 1)$$

which may be written

$$S^5 - 11S = \pm (S^4 - 3S^2 - 1)\sqrt{11}.$$

Since the quantity in the parenthesis is negative for $S = 2\sin\pi/11$, the positive sign must be chosen. This gives the result

$$S^5 - (S^4 - 3S^2 - 1)\sqrt{11} - 11S = 0.$$

Also solved by T. M. Blakslee, J. S. Brown, J. B. Faught, and A. V. Richardson.

2866 [1920, 482]. Proposed by NORMAN ANNING, University of Michigan.

Equilateral triangles whose sides are 1, 3, 5, 7, $\cdots$ are placed so that their bases lie corner to corner in a straight line. Show that the vertices lie upon a parabola and are all at integral distances from its focus.

Solution by G. W. Smith, University of Kansas.

If the base line of the triangles is taken as the x-axis and the y-axis is made to pass through the vertex of the first triangle of the series, the coördinates of the vertices of the triangles are $(0, \pm\frac{1}{2}\sqrt{3})$, $(2, \pm\frac{3}{2}\sqrt{3})$, $(6, \pm\frac{5}{2}\sqrt{3})$, $\cdots$. The vertex of the nth triangle is given by the equations $x = n(n-1)$, $y = \pm[(2n-1)/2]\sqrt{3}$. Eliminating n from these, gives $y^2 = 3(x + \frac{1}{4})$, a parabola whose focus is $(\frac{1}{2}, 0)$. The distance from the vertex $[n^2 - n, [(2n-1)/2]\sqrt{3}]$ to $(\frac{1}{2}, 0)$ is $n^2 - n + 1$ which is integral. It is worth noticing above that the second difference in the abscissas of the vertices is constant while it is the first difference in the ordinates that is constant, a property of the rational integral function of the second degree, such as x is of y.

Also solved by W. E. Anderson, T. L. Bennett, Augustus Bogard, H. C. Bradley, E. F. Canaday, P. J. da Cunha, J. F. Dove, L. H. Dube, W. C. Eells, J. B. Faught, Michael Goldberg, A. M. Harding, Lesta Hoel, G. R. Livingston, Gertrude I. McCain, I. Maizlish, E. W. Martin, L. C. Mathewson, H. L. Olson, Arthur Pelletier, R. V. Pritchard, R. V. Richardson, and T. O. Walton.

[1] See Burnside and Panton, *The Theory of Equations*, eighth edition, vol. 1, 1918, p. 90 where the necessary reductions are given; on page 100, ex. 3 the resulting equation (2) is also given.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will coöperate in contributing to the general interest of this department by sending items to the Editor-in-Chief.

At Stevens Institute of Technology Mr. W. E. F. APPUHN has been promoted to the rank of assistant professor.

At Davis and Elkins College, Mr. H. W. WHETSELL has been promoted from an instructorship to a professorship of mathematics.

At Sweet Briar College, Miss EUGENIE M. MORENUS has been appointed professor of mathematics and Miss MARY SEARLE instructor.

At Milwaukee-Downer College, Miss FRANCES A. ATWATER has been appointed instructor in mathematics and head of the department.

At the University of Georgia, Mr. A. H. STEVENS has been appointed instructor in mathematics.

At Colgate University, Mr. H. A. D. BELL has been appointed instructor in mathematics.

At Hope College, Holland, Mich., Mr. CLARENCE KLEIS has been appointed assistant in mathematics.

At Denison University, Mr. R. A. SHEETS has been promoted from instructor to assistant professor of mathematics and Mr. H. B. LEMON has been appointed instructor.

At the University of North Dakota, Professor R. R. HITCHCOCK, head of the department of mathematics, is on leave of absence pursuing studies in Columbia University. Associate Professor R. F. CASTNER has been appointed acting head of the department and J. D. LEITH instructor.

At the University of Wisconsin, Mr. M. L. MACQUEEN and Miss CAMILLA HAYDEN have been appointed instructors in mathematics. Mr. L. H. BUNYAN, Mr. R. H. MARQUIS, Mr. W. F. ARMENTROUT and Mr. D. W. MCLENEGAN have been appointed assistants.

At Hunter College, Miss EVELYN WALKER has been appointed assistant professor of mathematics.

At the Georgia School of Technology, Mr. W. V. SKILES has been promoted from an associate professorship of mathematics to a professorship. Mr. D. M. SMITH and Mr. A. B. MORTON have been promoted from assistant professorships to associate professorships and Mr. G. T. TRAWICH has been appointed instructor.

At Randolph-Macon Woman's College, Lynchburg, Va., Miss GILLIE A. LAREW (*1921*, 236) has been promoted from associate professor of mathematics to professor. Miss M. K. BOWEN, instructor, is on leave of absence pursuing graduate studies at the University of Chicago. Her place being temporarily filled by Miss VIRGINIA WATTS (*1921*, 236).

At Beloit College, Mr. H. H. CONWELL (*1921*, 43) has been promoted from an associate professorship of mathematics to a professorship and Mr. R. M. ROBINSON has been appointed instructor.

At the Municipal University of Akron, O., Miss W. H. LIPSCOMBE has been appointed instructor of mathematics.

At the University of Montana, Professor N. J. LENNES is on leave of absence for the current academic year. Miss GERTRUDE CLARK has been appointed assistant in mathematics.

At Ottawa University, Mr. H. W. BAILEY has been appointed instructor of mathematics.

At Tufts College Mr. P. D. WILKINS has been appointed instructor of mathematics.

At the University of Rochester, Professor A. S. GALE is serving in the newly created office of freshmen dean in addition to his duties as professor of mathematics and Mr. T. R. LONG has been appointed assistant in mathematics.

At Lake Forest College, Miss MARIE M. JOHNSON has been appointed instructor of mathematics.

At Creighton University, Mr. DAVID HICHES has been appointed professor of mathematics.

At Kenyon College, Mr. F. L. WHITE has been appointed instructor of mathematics.

At Miami University, Professor A. E. YOUNG has resigned to take a position with the Standard Oil Company in Pittsburgh, Pa. Professor W. E. ANDERSON has been promoted from an associate professorship of mathematics to a professorship and made head of the department. Assistant Professor J. C. BEEKLEY has resigned to pursue studies in physical chemistry at Princeton University. Mr. G. W. SPENCELY and Mr. C. D. EHRMAN (*1921*, 42) have been appointed to assistant professorships in mathematics.

At Union College, Mr. LAW BOWMAN, instructor in mathematics, has resigned.

At Lehigh University, Mr. H. S. BUNN has been appointed instructor in mathematics.

At Allegheny College, Mr. R. H. SKELTON has been appointed assistant professor of applied mathematics to succeed Professor K. A. MILLER.

At Cornell College, Mount Vernon, Iowa, Mr. EDMUND INGALLS has been appointed instructor in mathematics.

At the University of Maine, Messrs. F. S. BEALE, W. E. LORING, and EDWARD BROW have been appointed instructors. Mr. M. F. JORDAN, instructor in mathematics, has resigned to accept a graduate fellowship in astronomy at Harvard University. Mr. A. S. ADAMS, instructor, has resigned to become principal of eastern Maine Conference Seminary, and Mr. T. H. JOHNSON, instructor, has accepted a position in mathematics at the Moses Brown School.

At Syracuse University, Professor W. H. METZLER, head of the department of mathematics, has been made dean of the College of Liberal Arts.

At the University of Pittsburgh, Mr. Z. N. HOLLER has been appointed instructor in mathematics. Mr. PAUL FRANK and Mr. W. J. BRAZLER, instructors in mathematics, have resigned.

At Carleton College, Dr. C. H. GINGRICH of the department of mathematics and astronomy is on leave of absence for the current year pursuing work at the Mt. Wilson Observatory.

From June 20 to August 9, 1921, Professor E. R. HEDRICK, of the University of Missouri, delivered addresses, on behalf of the National Committee on Mathematical Requirements, at the following Institutions: University of Texas, University of Oklahoma, University of Nebraska, State Normal Schools at Peru and Kearney (Nebraska), University of Chicago, University of Iowa, Iowa State Teachers' College, University of Michigan, and Northwestern University.

Associate Professor EMMA KONANTZ, of Ohio Wesleyan University, who has spent the last two years in China, has in addition to teaching in Peking University, been "studying the history of Chinese mathematics, and assisting Dr. C. T. Chen, head of the department of mathematics in Peking University, in the preparation of a translation with explanatory notes, of a Chinese algebra of the thirteenth century. This algebra shows the solution of equations of higher degree, in four unknown quantities, and contains processes approximately the same as Horner's method of root extraction, the Chinese thus antedating Western mathematics several centuries."—*Ohio Wesleyan Alumni Quarterly.*

At the meeting in Boston, November 9, 1921, of the American Academy of Arts and Sciences, A. G. WEBSTER, of Clark University, delivered an address on "Hermann von Helmholtz and his significance for a century of science."

In the meetings of the American Mathematical Society at Columbia University, October 29, 1921, 11 papers were presented by the following members: G. A. CAMPBELL, T. H. GRONWALL, EDWARD KASNER, R. L. MOORE (2), L. H. RICE, J. F. RITT (2), L. B. ROBINSON, J. L. WALSH, and J. K. WHITTEMORE. Details concerning the papers may be found in *Bulletin of the American Mathematical Society*, March, 1922.

At the nineteenth annual meeting of the Association of Teachers of Mathematics in New England held at Massachusetts Institute of Technology on December 3, 1921, the following papers were read: "Geometry understood, not memorized" by R. R. SMITH of the Newton Classical High School; "Modern methods in junior high schools" by E. R. SMITH of Park School, Baltimore, Md.; "Graphic solutions of certain systems of equations" by R. E. BRUCE of Boston University.

At the fall meeting of the Connecticut Valley Section of the Association of Teachers of Mathematics in New England, at Hartford, November 5, 1921, the following papers were presented: "An introduction to trigonometry" by H. E. WEBB; "The work of the examiners of the College Entrance Examination Board" by A. E. BOOTH; and "The new statement of college entrance requirements in mathematics" by W. R. LONGLEY. Professor P. F. SMITH is president of the Section, Mr. M. M. S. MORIARTY, vice-president, Professor ELEANOR C. DOAK, treasurer, and Mr. H. B. MARSH a director.

The mathematics department of the Virginia Educational Conference met in annual session, November 23, 1921, in John Marshall High School, Richmond, Va.

The following papers were read: "Fractions is fractions" by E. G. SCHAKELFORD; "An experiment in classroom method" by EVELYN WHITMORE; "The coördination of high school and first year college mathematics" by J. J. LUCK and J. D. RIDDICK; "The practical use of mathematics" by J. E. ROWE. The following officers were reëlected: T. McN. SIMPSON, Jr., president, and C. W. GIVENS, secretary.

At the Toronto meetings of the American Mathematical Society, December 28–29, 1921, 32 papers by the following mathematicians were presented: S. BEATTY, R. W. BURGESS, W. L. CRUM, LOUISE D. CUMMINGS, D. R. CURTISS, E. L. DODD, L. P. EISENHART and OSWALD VEBLEN, ARNOLD EMCH, H. S. EVERETT, C. H. FORSYTH, J. S. C. GLASHAN (2), O. E. GLENN, C. F. GUMMER, OLIVE C. HAZLETT, EINAR HILLE, E. V. HUNTINGTON, W. A. HURWITZ, M. H. INGRAHAM, JOSEPH LIPKA, H. F. MACNEISH, G. A. MILLER (2), E. H. MOORE, I. R. POUNDER, I. J. SCHWATT (2), J. L. WALSH and NORBERT WIENER, E. J. WILCZYNSKI, C. E. WILDER (2), and S. D. ZELDIN. At the joint session with the American Physical Society and with Sections B and C of the American Association for the Advancement of Science, the American Physical Society was represented by SAUL DUSHMAN, Sections B and C by J. C. MCLENNAN and R. C. TOLMAN respectively, and the American Mathematical Society by H. B. PHILLIPS. At the joint session with the Mathematical Association of America and with Section A of the Association for the Advancement of Science addresses were made by R. D. CARMICHAEL, D. R. CURTISS, H. E. SLAUGHT, and R. M. YERKES. Details concerning these meetings may be found in this MONTHLY and in the *Bulletin of the American Mathematical Society*.

Published August 1, 1922.

SIXTH ANNUAL MEETING OF THE MATHEMATICAL ASSOCIATION OF AMERICA.

By invitation of the University of Toronto and the Royal Canadian Institute, the sixth annual meeting of the Mathematical Association of America was held at the University of Toronto on Thursday and Friday, December 29 and 30, 1921, in affiliation with the American Association for the Advancement of Science and in connection with the annual meeting of the American Mathematical Society. There were 110 in attendance at the sessions, including the following 88 members of the Association:

R. C. ARCHIBALD, Brown University.
G. N. ARMSTRONG, Ohio Wesleyan University.
C. S. ATCHISON, Washington and Jefferson College.
CLARA L. BACON, Goucher College.
L. A. BAUER, Dept. of Terrestrial Magnetism, Washington, D. C.
SAMUEL BEATTY, University of Toronto.
R. D. BEETLE, Dartmouth College.
O. F. H. BERT, Washington and Jefferson College.
G. A. BLISS, University of Chicago.
J. W. BRADSHAW, University of Michigan.
R. W. BURGESS, Brown University.
W. D. CAIRNS, Oberlin College.
H. C. CARVER, University of Michigan.
LOUISE D. CUMMINGS, Vassar College.
C. H. CURRIER, Brown University.
D. R. CURTISS, Northwestern University.
A. T. DE LURY, University of Toronto.
W. W. DENTON, University of Michigan.
Sister MARIOLA DOBBIN, St. Clara College.
B. F. DOSTAL, University of Denver.
L. H. DUBE, Ottawa University.
ARNOLD DRESDEN, University of Wisconsin.
PETER FIELD, University of Michigan.
J. C. FIELDS, University of Toronto.
WILLIAM FINDLAY, McMaster University.
B. F. FINKEL, Drury College.
T. M. FOCKE, Case School of Applied Science.
W. B. FORD, University of Michigan.
J. L. GIBSON, University of Utah.
D. C. GILLESPIE, Cornell University.
O. E. GLENN, University of Pennsylvania.
J. W. GLOVER, University of Michigan.
R. A. GRAY, Oakwood Collegiate Institute, Toronto.
C. F. GUMMER, Queen's University.
OLIVE C. HAZLETT, Mt. Holyoke College.
E. R. HEDRICK, University of Missouri.
H. A. HOWE, University of Denver.
E. V. HUNTINGTON, Harvard University.
W. A. HURWITZ, Cornell University.
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N. J. LENNES, University of Montana.
FLORENCE P. LEWIS, Goucher College.
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NORMAN MILLER, Queen's University.
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E. H. MOORE, University of Chicago.
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F. W. OWENS, Cornell University.
Mrs. F. W. OWENS, Cornell University.
W. J. PATTERSON, Western University.
Mrs. ANNA J. PELL, Bryn Mawr College.
H. B. PHILLIPS, Mass. Institute of Technology.
A. D. PITCHER, Western Reserve University.
I. R. POUNDER, University of Toronto.
P. L. REA, Marietta College.
A. V. RICHARDSON, Bishop's College.
R. G. D. RICHARDSON, Brown University.
H. L. RIETZ, University of Iowa.
G. M. ROBISON, Cornell University.
E. D. ROE, Jr., Syracuse University.
W. H. ROEVER, Washington University.
N. E. SHEPPARD, University of Toronto.
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CLARA E. SMITH, Wellesley College.
E. R. SMITH, State College of Iowa.
I. W. SMITH, North Dakota Agric. College.